Normalized Completion for Stratified Linear Rewriting Systems

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1. Introduction
2. Stratified rewriting systems
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Contents

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- **Algebraic rewriting** is a rewriting theory for the presentation of algebraic structures, aimed at solving

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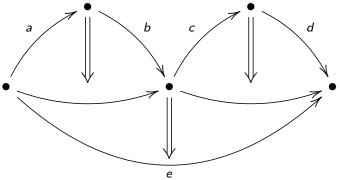
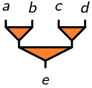
- **Computing linear bases by confluence**
  - Every element of a terminating and confluent rewriting system admits a unique normal form.
  - These normal forms provide canonical representatives.
  - The set of normal forms yields a **linear basis** of the presented algebraic structure.

Diagrammatic rewriting

- **Diagrammatic rewriting** is a rewriting model for **linear 2-categories**, in which 2-cells are represented by 2-dimensional diagrams, such as





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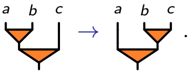
Diagrammatic rewriting

- **Example (Associative category).** The associative theory is presented by the relation

$$(a \star_0 b) \star_0 c = a \star_0 (b \star_0 c).$$

Diagrammatically, this relation can be oriented as the rewriting rule on 2-cells



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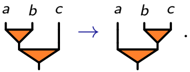

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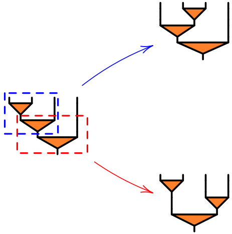

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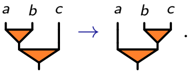
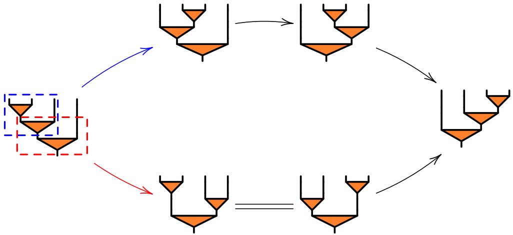


- Overlapping applications of this rule naturally give rise to a critical branching, which is confluent:



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    - **Complex termination functions** (e.g. KLR algebras);
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  - Two-dimensional rewriting involves **complex interactions between rules**.
- **Our objective**
  Develop completion procedures and termination methods adapted to diagrammatic rewriting by stratification.

Ideas

- **Stratified normalisation completion modulo**
  - Rewriting modulo (Huet '80, Jouannaud-Kirchner '84);
  - Normalisation and completion modulo (Marche '96);
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- **Question**

  Can one construct a **Tietze equivalent** rewriting system (X, R') endowed with a stratification for which these properties become modular?

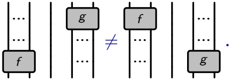
Contents

Stratified rewriting systems

- A **linear 2-precategory** $\mathcal{C}$ is a linear 2-category consisting of 0-cells, 1-cells, and 2-cells, where 2-cells can be composed both horizontally and vertically, such that
  - The **interchange laws** are not imposed, that is, the vertical composition of 2-cells is not required to be functorial:

$$(f \circ_0 id \circ_0 s_1(g)) \circ_1 (t_1(f) \circ_0 id \circ_0 g) \neq (s_1(f) \circ_0 id \circ_0 g) \circ_1 (f \circ_0 id \circ_0 t_1(g)).$$

  Diagrammatically,



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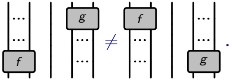
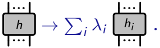


- A **rewriting rule** on $\mathcal{C}$ is of the form



A **stratification** of length k of the set R of rewriting rules is a partition

$$R = F_s^1 R \sqcup F_s^2 R \sqcup \cdots \sqcup F_s^k R,$$

denoted by $F_s^* R$. For every $1 \leq i \leq k$, set

$$F^i R = F_s^1 R \sqcup \cdots \sqcup F_s^i R.$$

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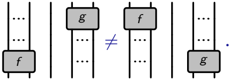

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- A set R of rewriting rules equipped with such a stratification is called a ***k*-stratified rewriting system**.

Stratified termination functions

- A **weighted function** on a precategory $\mathcal{C}$ is a hom-set–wise function

$$\delta : \mathcal{C} \longrightarrow (Z, \prec),$$

such that

$$\delta(f) \prec \delta(g) \quad \text{implies} \quad \delta(C[f]) \prec \delta(C[g]),$$

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- A **termination function** of a k-stratified rewriting system $F_s^* R$ is a family of weighted functions $(\delta_i)_{1 \leq i \leq k}$, such that

$$\begin{cases} \delta_i(t\alpha) < \delta_i(s\alpha), & \alpha \in F_s^i R, \\ \delta_i(t\beta) \leq \delta_i(s\beta), & \beta \in F^{i-1} R. \end{cases}$$

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- **Theorem.** If a k-stratified rewriting system $F_s^* R$ admits a termination function, then it is terminating.

- **Remark.** Stratified termination functions generalize the derivation approach introduced by Guiraud–Malbos '09, which is widely used in diagrammatic rewriting.

Contents

Normalisation

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Step 2. Normalise every $\alpha \in F_s^2 R$ in each admissible context C to define a new rule $C[\alpha]\!\downarrow$:

$$\begin{tikzcd} C[s\alpha] \arrow[r, "C[\alpha]"] \arrow[d, twoheadrightarrow] & C[t\alpha] \arrow[d, twoheadrightarrow] \\ C[s\alpha]\!\downarrow \arrow[r, leftrightarrow, dashed, "C[\alpha]\downarrow"] & C[t\alpha]\!\downarrow \end{tikzcd}$$

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$$[\![\alpha]\!]\!\downarrow := \{C[\alpha]\!\downarrow\}. \qquad \text{and} \qquad F_s^2 \widehat{R} := \bigsqcup_{\alpha \in F_s^2 R} \min_{\sqsubseteq} [\![\alpha]\!]\!\downarrow,$$

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Iteration. Continue recursively to define $F_s^i \widehat{R}$ for all $3 \leq i \leq k$.

Main results

- The **confluence** of the normalisation $\widehat{R}$ is **modular**:

  Theorem (Modularity of stratified rewriting).
  If every stratum $F_s^i\widehat{R}$ is confluent for $1 \leq i \leq k$, then their union $\widehat{R} = F^k\widehat{R}$ is confluent.

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- **Corollary.** Let R be a stratified rewriting system presenting a linear monoidal category $\mathcal{C}$. Then the set $\mathrm{Nf}_m(\widehat{R})$ forms a linear hom-basis of $\mathcal{C}$.

Example

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$$F_s^1 \widehat{R} := F_s^1 R, \qquad \mathrm{Nf}(F^1 \widehat{R}) = \{ z^i y^j x^k \mid i, j, k \geq 0 \}.$$

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$$C[\alpha] : \boldsymbol{z}^{\boldsymbol{i}} \boldsymbol{y}^{\boldsymbol{j}} \boldsymbol{x}^{\boldsymbol{k}} \, \boldsymbol{xyz} \, \boldsymbol{z}^{\boldsymbol{p}} \boldsymbol{y}^{\boldsymbol{q}} \boldsymbol{x}^{\boldsymbol{r}} \quad \to \quad \boldsymbol{z}^{\boldsymbol{i}} \boldsymbol{y}^{\boldsymbol{j}} \boldsymbol{x}^{\boldsymbol{k}} \, (\boldsymbol{x}^2 + \boldsymbol{y}^2 + \boldsymbol{z}^2) \, \boldsymbol{z}^{\boldsymbol{p}} \boldsymbol{y}^{\boldsymbol{q}} \boldsymbol{x}^{\boldsymbol{r}}.$$

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$$C[\alpha]\!\downarrow\colon \boldsymbol{z^{i+p+1} y^{j+q+1} x^{k+r+1}} \to \boldsymbol{z^{i+p} y^{j+q} x^{k+r+2}} + \boldsymbol{z^{i+p} y^{j+q+2} x^{k+r}} + \boldsymbol{z^{i+p+2} y^{j+q} x^{k+r}}.$$

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Goal. Compute a linear basis of the presented algebra by stratification.

Input. A 2-stratified rewriting system $F_s^* R$ with generators x, y, z and

$$F_s^1 R = \left\{ \begin{array}{ccc} \boldsymbol{xy} & \to & \boldsymbol{yx}, \\ \boldsymbol{yz} & \to & \boldsymbol{zy}, \\ \boldsymbol{xz} & \to & \boldsymbol{zx} \end{array} \right\}, \qquad F_s^2 R = \left\{ \alpha : \boldsymbol{xyz} \to \boldsymbol{x}^2 + \boldsymbol{y}^2 + \boldsymbol{z}^2 \right\}.$$

Step 1. Let $\preccurlyeq$ be the lexicographic order with $z \prec y \prec x$. The first stratum is confluent:

$$F_s^1 \widehat{R} := F_s^1 R, \qquad \mathrm{Nf}(F^1 \widehat{R}) = \{ z^i y^j x^k \mid i, j, k \geq 0 \}.$$

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$$F_s^2 \widehat{R} := \left\{ zy^{n+1}x \overset{C[\alpha]\downarrow}{\rightarrow} y^n x^2 + y^{n+2} + z^2 y^n \;\middle|\; n \geq 0 \right\}.$$

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Conclusion.
The stratum $F_s^2 \widehat{R}$ has no critical branching. Hence $\widehat{R}$ is confluent. Therefore, the presented algebra admits the linear basis

$$\mathrm{Nf}_m(\widehat{R}) = \{ z^i x^j, \; y^i x^j, \; z^i y^j \mid i, j \geq 0 \}.$$

Stratified normalisation and Knuth–Bendix completion

Combining the stratified normalisation procedure with the Knuth-Bendix completion procedure yields the following **completion procedure for stratified rewriting systems**.

$$\underbrace{F_s^1 \quad \sqcup \quad F_s^2 \quad \sqcup \quad \cdots \quad \sqcup \quad F_s^k}_{\text{SN}}$$

$$\begin{tikzcd}
\widehat{F_s^1} \arrow[d, "\text{KB}"'] & \sqcup & \widehat{F_s^2} \arrow[d, "\text{KB}"] & \sqcup & \cdots & \sqcup & \widehat{F_s^k} \arrow[d, "\text{KB}"] \\
\widetilde{\widehat{F_s^1}} & \sqcup & \widetilde{\widehat{F_s^2}} & \sqcup & \cdots & \sqcup & \widetilde{\widehat{F_s^k}}
\end{tikzcd}$$

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Contents

Filtration of admissible contexts

- For **string rewriting systems**, admissible contexts can often be described recursively.

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- For **diagrammatic rewriting systems**, however, admissible contexts are typically much harder to compute explicitly.

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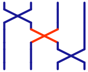

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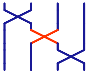

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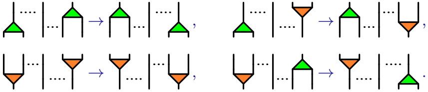
- Hence, we introduce a technical method to deal with these complicated contexts according to their **length**.

Example

Input. A 2-category $\mathcal{C}$ presented by a 2-stratified rewriting system $F_s^* R$.
The first stratum $F_s^1 R$ consists of



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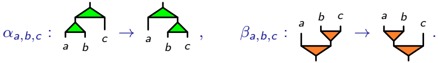

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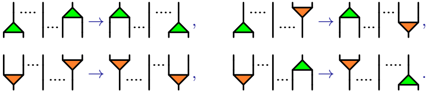

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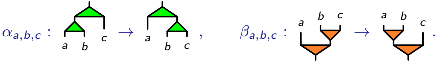

Contexts of length 0. Set $F_s^1 \widehat{R} = F_s^1 R$. Since $C[\alpha]{\downarrow} = \alpha$ for contexts C of length 0, we obtain $\min_{\sqsubseteq} [\![\alpha]\!]{\downarrow}_1^0 = \{\alpha\}$.

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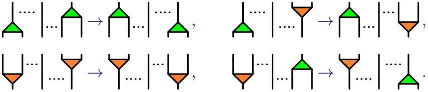

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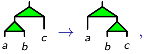
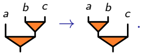
$\alpha_{a,b,c}$: 

, $\beta_{a,b,c}$: 

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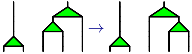
$C[\alpha]$: 

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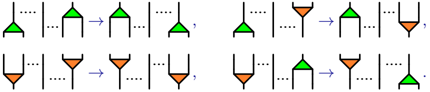

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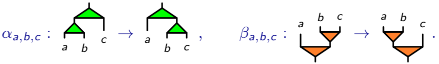

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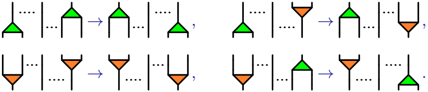

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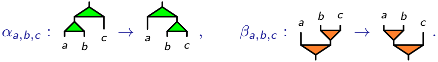

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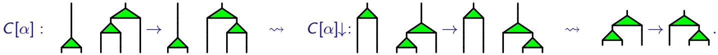

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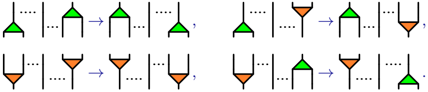

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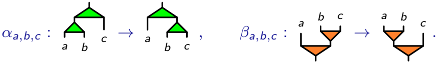

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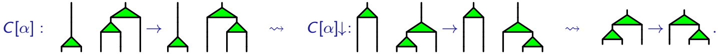


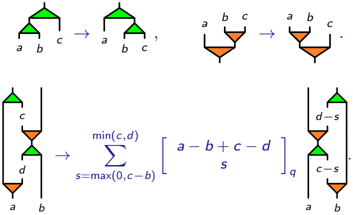
Conclusion. Hence

$$\min_{\sqsubseteq} [\![\alpha]\!]{\downarrow}_1^1 = \{\alpha\} = \min_{\sqsubseteq} [\![\alpha]\!]{\downarrow}_1^0, \qquad F_s^2 \widehat{R} = F_s^2 R.$$

Since $F_s^1 \widehat{R}$ and $F_s^2 \widehat{R}$ are confluent, $\mathrm{Nf}_m(\widehat{R})$ forms a hom-basis of $\mathcal{C}$.

Normal form basis of q-Schur category by stratification

- In Brundan '2025, a presentation of the q-Schur category is given:



for all $a, b, c, d \geq 0$ with $d \leq a$ and $c \leq b + d$.

- From this presentation, we construct a 5-stratified rewriting system $F_s^*\mathrm{qS}$, compute its normalisation $\widehat{\mathrm{qS}}$ of the q-Schur category qS.

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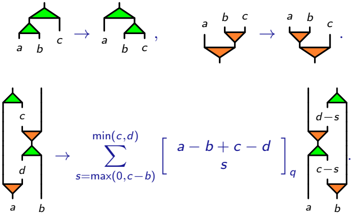

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- **Theorem.** The set $\mathrm{Nf}(\widehat{\mathrm{qS}})$ forms a hom-basis of the q-Schur category.

- This hom-basis, called the **normal form basis**, differs from the one obtained in Brundan '25.

Contents

Future work

- Develop a stratified normalisation procedure for **rewriting modulo**, with applications to:
  - Affine oriented Brauer categories Brundan-Comes-Nash-Reynolds '17;
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- Improve stratified normalisation procedures through:
  - **Filtration** of admissible contexts according to their length;
  - **Automata descriptions** of admissible contexts.

Thank you for your attention.

Institut Camille Jordan

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