Enumerating Ground Canonical Rewrite Systems

Masahiko Sakai
Nagoya University

Aart Middeldorp
University of Innsbruck

Sarah Winkler
Free University of Bozen / Bolzano

Outline

JSC 1993

J. Symbolic Computation (1993) **15,** 415–450

A Fast Algorithm for Generating Reduced Ground Rewriting Systems from a Set of Ground Equations

WAYNE SNYDER

Department of Computer Science
Boston University
Boston, MA 02215
snyder@cs.bu.edu

In this paper we give a fast method for generating reduced sets of rewrite rules equivalent to a given set of ground equations. Since reduced sets of ground rewrite rules are in fact canonical, this is an efficient Knuth-Bendix procedure for the ground case. The dominant cost of the algorithm is for congruence closure, so the method runs in $O(n\,log\,n)$ time and quadratic space, or in $O(n\,(log\,n)^2)$ time and linear space, where n is the number of occurrences of symbols in the original set of ground equations E. We also show how our method provides a precise characterization of the (finite) collection of all reduced sets of rewrite rules equivalent to a given ground set of equations E, and prove that our algorithm is complete in the sense that it can enumerate every member of this collection. Finally, we show that a modified version of this procedure can produce a reduced ground rewriting system contained in the lexicographic path ordering generated by a given total precedence ordering on the symbols of E, still in a worst-case time of $O(n\,log\,n)$.

1 Introduction

The theory of ground equality systems has developed in the theorem proving community in two
ra... distin... nds. ... researchers [14, 26, 31, 36] studied wor...

JSC 1993

J. Symbolic Computation (1993) **15**, 415–450

A Fast Algorithm for Generating Reduced Ground Rewriting Systems from a Set of Ground Equations

WAYNE SNYDER

Department of Computer Science
Boston University
Boston, MA 02215
snyder@cs.bu.edu

In this paper we give a fast method for generating reduced sets of rewrite rules equivalent to a given set of ground equations. Since reduced sets of ground rewrite rules are in fact canonical, this is an efficient Knuth-Bendix procedure for the ground case. The dominant cost of the algorithm is for congruence closure, so the method runs in $O(n \log n)$ time and quadratic space, or in $O(n (\log n)^2)$ time and linear space, where n is the number of occurrences of symbols in the original set of ground equations E. We also show how our method provides a precise characterization of the (finite) collection of all reduced sets of rewrite rules equivalent to a given ground set of equations E, and prove that our algorithm is complete in the sense that it can enumerate every member of this collection. Finally, we show that a modified version of this procedure can produce a reduced ground rewriting system contained in the lexicographic path ordering generated by a given total precedence ordering on the symbols of E, still in a worst-case time of $O(n \log n)$.

1 Introduction

The theory of ground equality systems has developed in the theorem proving community in two ...

Definitions

TRS $\mathcal{R}$ is

- right–reduced if $r \in \mathsf{NF}(\mathcal{R})$ for every rewrite rule $\ell \to r$ in $\mathcal{R}$

Definitions

TRS $\mathcal{R}$ is

- right–reduced if $r \in \mathsf{NF}(\mathcal{R})$ for every rewrite rule $\ell \to r$ in $\mathcal{R}$
- left–reduced if $\ell \in \mathsf{NF}(\mathcal{R} \setminus \{\ell \to r\})$ for every rewrite rule $\ell \to r$ in $\mathcal{R}$

Definitions

TRS $\mathcal{R}$ is

- right–reduced if $r \in \mathsf{NF}(\mathcal{R})$ for every rewrite rule $\ell \to r$ in $\mathcal{R}$
- left–reduced if $\ell \in \mathsf{NF}(\mathcal{R} \setminus \{\ell \to r\})$ for every rewrite rule $\ell \to r$ in $\mathcal{R}$
- reduced if it is both left–reduced and right–reduced

Definitions

TRS $\mathcal{R}$ is

- right–reduced if $r \in \mathsf{NF}(\mathcal{R})$ for every rewrite rule $\ell \to r$ in $\mathcal{R}$
- left–reduced if $\ell \in \mathsf{NF}(\mathcal{R} \setminus \{\ell \to r\})$ for every rewrite rule $\ell \to r$ in $\mathcal{R}$
- reduced if it is both left–reduced and right–reduced
- canonical if it is confluent, terminating and reduced

Example

three TRSs

$$\begin{aligned} f(a) &\to g(b,b) \\ h(a) &\to b \\ g(b,b) &\to a \\ i(a) &\to c \end{aligned}$$

1

$$\begin{aligned} f(a) &\to a \\ b &\to h(a) \\ g(h(a),h(a)) &\to a \\ c &\to i(a) \end{aligned}$$

2

$$\begin{aligned} f(g(b,b)) &\to g(b,b) \\ a &\to g(b,b) \\ h(g(b,b)) &\to b \\ i(f(a)) &\to c \end{aligned}$$

3

Example

three TRSs

$$\begin{aligned} f(a) &\to g(b,b) \\ h(a) &\to b \\ g(b,b) &\to a \\ i(a) &\to c \end{aligned}$$

1

$$\begin{aligned} f(a) &\to a \\ b &\to h(a) \\ g(h(a),h(a)) &\to a \\ c &\to i(a) \end{aligned}$$

2

$$\begin{aligned} f(g(b,b)) &\to g(b,b) \\ a &\to g(b,b) \\ h(g(b,b)) &\to b \\ i(f(a)) &\to c \end{aligned}$$

3

not right−reduced

Example

three TRSs

$$\begin{aligned} f(a) &\to g(b,b) \\ h(a) &\to b \\ g(b,b) &\to a \\ i(a) &\to c \end{aligned}$$

1

not right−reduced

$$\begin{aligned} f(a) &\to a \\ b &\to h(a) \\ g(h(a),h(a)) &\to a \\ c &\to i(a) \end{aligned}$$

2

canonical

$$\begin{aligned} f(g(b,b)) &\to g(b,b) \\ a &\to g(b,b) \\ h(g(b,b)) &\to b \\ i(f(a)) &\to c \end{aligned}$$

3

Example

three TRSs

$$\begin{aligned} f(a) &\to g(b,b) \\ h(a) &\to b \\ g(b,b) &\to a \\ i(a) &\to c \end{aligned}$$

1

not right−reduced

$$\begin{aligned} f(a) &\to a \\ b &\to h(a) \\ g(h(a),h(a)) &\to a \\ c &\to i(a) \end{aligned}$$

2

canonical

$$\begin{aligned} f(g(b,b)) &\to g(b,b) \\ a &\to g(b,b) \\ h(g(b,b)) &\to b \\ i(f(a)) &\to c \end{aligned}$$

3

not left−reduced

Definitions

TRS $\mathcal{R}$ is

- right–reduced if $r \in \mathsf{NF}(\mathcal{R})$ for every rewrite rule $\ell \to r$ in $\mathcal{R}$
- left–reduced if $\ell \in \mathsf{NF}(\mathcal{R} \setminus \{\ell \to r\})$ for every rewrite rule $\ell \to r$ in $\mathcal{R}$
- reduced if it is both left–reduced and right–reduced
- canonical if it is confluent, terminating and reduced

Theorem (Snyder 1993)

reduced ground TRSs are canonical

Definitions

TRS $\mathcal{R}$ is

- right–reduced if $r \in \mathrm{NF}(\mathcal{R})$ for every rewrite rule $\ell \to r$ in $\mathcal{R}$
- left–reduced if $\ell \in \mathrm{NF}(\mathcal{R} \setminus \{\ell \to r\})$ for every rewrite rule $\ell \to r$ in $\mathcal{R}$
- reduced if it is both left–reduced and right–reduced
- canonical if it is confluent, terminating and reduced

Theorem (Snyder 1993)

reduced ground TRSs are canonical

Definition

TRSs $\mathcal{R}$ and $\mathcal{S}$ are equivalent if $\leftrightarrow_{\mathcal{R}}^{*} = \leftrightarrow_{\mathcal{S}}^{*}$

Known Results

① congruence closure is efficient technique for solving word problems for ground ESs

Known Results

① congruence closure is efficient technique for solving word problems for ground ESs

② completion can be tamed for ground ESs such that it always terminates in canonical TRS

Known Results

① congruence closure is efficient technique for solving word problems for ground ESs

② completion can be tamed for ground ESs such that it always terminates in canonical TRS

③ every canonical presentation of ground ES can be obtained by ground completion

Known Results

① congruence closure is efficient technique for solving word problems for ground ESs

② completion can be tamed for ground ESs such that it always terminates in canonical TRS

③ every canonical presentation of ground ES can be obtained by ground completion

④ number of different canonical presentations for ground ES consisting of k equations is $\leqslant 2^k$

Known Results

① congruence closure is efficient technique for solving word problems for ground ESs

② completion can be tamed for ground ESs such that it always terminates in canonical TRS

③ every canonical presentation of ground ES can be obtained by ground completion

④ number of different canonical presentations for ground ES consisting of k equations is $\leqslant 2^k$

⑤ first–order theory of rewriting is decidable for ground TRSs (Dauchet and Tison, LICS 1990)

Known Results

① congruence closure is efficient technique for solving word problems for ground ESs

② completion can be tamed for ground ESs such that it always terminates in canonical TRS

③ every canonical presentation of ground ES can be obtained by ground completion

④ number of different canonical presentations for ground ES consisting of k equations is $\leqslant 2^k$

⑤ first−order theory of rewriting is decidable for ground TRSs (Dauchet and Tison, LICS 1990)

Remark

④ is based on detailed analysis of special structure−sharing dag

Outline

Notation

- $t\{\ell/r\}$ denotes term obtained from t after replacing all subterms r by ℓ

Notation

- $t\{\ell/r\}$ denotes term obtained from t after replacing all subterms r by ℓ
- $T\{\ell/r\} = \{t\{\ell/r\} \mid t \in T\}$ for set of terms T

Notation

- $t\{\ell/r\}$ denotes term obtained from t after replacing all subterms r by ℓ
- $T\{\ell/r\} = \{t\{\ell/r\} \mid t \in T\}$ for set of terms T
- $\mathcal{R}\{\ell/r\} = \{s\{\ell/r\} \to t\{\ell/r\} \mid s \to t \in \mathcal{R}\}$ for TRS $\mathcal{R}$

Notation

- $t\{\ell/r\}$ denotes term obtained from t after replacing all subterms r by ℓ
- $T\{\ell/r\} = \{t\{\ell/r\} \mid t \in T\}$ for set of terms T
- $\mathcal{R}\{\ell/r\} = \{s\{\ell/r\} \to t\{\ell/r\} \mid s \to t \in \mathcal{R}\}$ for TRS $\mathcal{R}$

Definition (Snyder's Transformation)

transformation

$$\mathcal{R} \uplus \{\ell \to r\} \quad \Longrightarrow \quad \mathcal{R}\{\ell/r\} \cup \{r \to \ell\} \tag{$\star$}$$

provided r is no (proper) subterm of ℓ

Notation

- $t\{\ell/r\}$ denotes term obtained from t after replacing all subterms r by ℓ
- $T\{\ell/r\} = \{t\{\ell/r\} \mid t \in T\}$ for set of terms T
- $\mathcal{R}\{\ell/r\} = \{s\{\ell/r\} \to t\{\ell/r\} \mid s \to t \in \mathcal{R}\}$ for TRS $\mathcal{R}$

Definition (Snyder's Transformation)

transformation

$$\mathcal{R} \uplus \{\ell \to r\} \quad \Longrightarrow \quad \mathcal{R}\{\ell/r\} \cup \{r \to \ell\} \tag{$\star$}$$

provided r is no (proper) subterm of ℓ (rule $\ell \to r$ is reversible)

Notation

- $t\{\ell/r\}$ denotes term obtained from t after replacing all subterms r by ℓ
- $T\{\ell/r\} = \{t\{\ell/r\} \mid t \in T\}$ for set of terms T
- $\mathcal{R}\{\ell/r\} = \{s\{\ell/r\} \to t\{\ell/r\} \mid s \to t \in \mathcal{R}\}$ for TRS $\mathcal{R}$

Definition (Snyder's Transformation)

transformation

$$\mathcal{R} \uplus \{\ell \to r\} \quad \Longrightarrow \quad \mathcal{R}\{\ell/r\} \cup \{r \to \ell\} \tag{$\star$}$$

provided r is no (proper) subterm of ℓ (rule $\ell \to r$ is reversible)

footnote (Snyder 1993)

The fact that this single transformation is sufficient to transform one reduced system into any other can be proved formally, but the proofs are tedious, and so we have preferred to present this intuitively.

Example (Snyder 1993)

$$\begin{aligned} f(a) &\approx g(b,b) & f(f(a)) &\approx a & f(f(f(a))) &\approx a \\ g(b,h(a)) &\approx g(b,b) & h(a) &\approx b & i(f(a)) &\approx c \end{aligned}$$

Example (Snyder 1993)

$$\begin{aligned} f(a) &\approx g(b,b) & f(f(a)) &\approx a & f(f(f(a))) &\approx a \\ g(b,h(a)) &\approx g(b,b) & h(a) &\approx b & i(f(a)) &\approx c \end{aligned}$$

admits six different canonical TRSs:

$$\begin{aligned} f(g(b,b)) &\xrightarrow{1} g(b,b) \\ a &\xrightarrow{2} g(b,b) \\ h(g(b,b)) &\xrightarrow{3} b \\ i(g(b,b)) &\xrightarrow{4} c \end{aligned}$$

Example (Snyder 1993)

$$\begin{aligned} \mathsf{f}(\mathsf{a}) &\approx \mathsf{g}(\mathsf{b},\mathsf{b}) & \mathsf{f}(\mathsf{f}(\mathsf{a})) &\approx \mathsf{a} & \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))) &\approx \mathsf{a} \\ \mathsf{g}(\mathsf{b},\mathsf{h}(\mathsf{a})) &\approx \mathsf{g}(\mathsf{b},\mathsf{b}) & \mathsf{h}(\mathsf{a}) &\approx \mathsf{b} & \mathsf{i}(\mathsf{f}(\mathsf{a})) &\approx \mathsf{c} \end{aligned}$$

admits six different canonical TRSs:

$$\begin{aligned} \mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) &\xrightarrow{1} \mathsf{g}(\mathsf{b},\mathsf{b}) \\ \mathsf{a} &\xrightarrow{2} \mathsf{g}(\mathsf{b},\mathsf{b}) \\ \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) &\xrightarrow{3} \mathsf{b} \\ \mathsf{i}(\mathsf{g}(\mathsf{b},\mathsf{b})) &\xrightarrow{4} \mathsf{c} \end{aligned} \quad \stackrel{2}{\Longrightarrow} \quad \begin{aligned} \mathsf{f}(\mathsf{a}) &\xrightarrow{1} \mathsf{a} \\ \mathsf{g}(\mathsf{b},\mathsf{b}) &\xrightarrow{2} \mathsf{a} \\ \mathsf{h}(\mathsf{a}) &\xrightarrow{3} \mathsf{b} \\ \mathsf{i}(\mathsf{a}) &\xrightarrow{4} \mathsf{c} \end{aligned}$$

Example (Snyder 1993)

$$\mathsf{f(a)} \approx \mathsf{g(b,b)} \qquad \mathsf{f(f(a))} \approx \mathsf{a} \qquad \mathsf{f(f(f(a)))} \approx \mathsf{a}$$
$$\mathsf{g(b,h(a))} \approx \mathsf{g(b,b)} \qquad \mathsf{h(a)} \approx \mathsf{b} \qquad \mathsf{i(f(a))} \approx \mathsf{c}$$

admits six different canonical TRSs:

$$\begin{array}{rcl} \mathsf{f(g(b,b))} & \xrightarrow{1} & \mathsf{g(b,b)} \\ \mathsf{a} & \xrightarrow{2} & \mathsf{g(b,b)} \\ \mathsf{h(g(b,b))} & \xrightarrow{3} & \mathsf{b} \\ \mathsf{i(g(b,b))} & \xrightarrow{4} & \mathsf{c} \end{array} \quad \overset{2}{\Longleftrightarrow} \quad \begin{array}{rcl} \mathsf{f(a)} & \xrightarrow{1} & \mathsf{a} \\ \mathsf{g(b,b)} & \xrightarrow{2} & \mathsf{a} \\ \mathsf{h(a)} & \xrightarrow{3} & \mathsf{b} \\ \mathsf{i(a)} & \xrightarrow{4} & \mathsf{c} \end{array}$$

Example (Snyder 1993)

$$f(a) \approx g(b,b) \qquad f(f(a)) \approx a \qquad f(f(f(a))) \approx a$$
$$g(b,h(a)) \approx g(b,b) \qquad h(a) \approx b \qquad i(f(a)) \approx c$$

admits six different canonical TRSs:

$$\begin{array}{rcl} f(g(b,b)) & \xrightarrow{1} & g(b,b) \\ a & \xrightarrow{2} & g(b,b) \\ h(g(b,b)) & \xrightarrow{3} & b \\ i(g(b,b)) & \xrightarrow{4} & c \end{array} \quad \overset{2}{\Longleftrightarrow} \quad \begin{array}{rcl} f(a) & \xrightarrow{1} & a \\ g(b,b) & \xrightarrow{2} & a \\ h(a) & \xrightarrow{3} & b \\ i(a) & \xrightarrow{4} & c \end{array}$$

$$\Updownarrow 4$$

$$\begin{array}{rcl} f(g(b,b)) & \xrightarrow{1} & g(b,b) \\ a & \xrightarrow{2} & g(b,b) \\ h(g(b,b)) & \xrightarrow{3} & b \\ c & \xrightarrow{4} & i(g(b,b)) \end{array}$$

Example (Snyder 1993)

$$
\begin{array}{rclcrclcrcl}
f(a) & \approx & g(b,b) & \quad & f(f(a)) & \approx & a & \quad & f(f(f(a))) & \approx & a \\
g(b,h(a)) & \approx & g(b,b) & & h(a) & \approx & b & & i(f(a)) & \approx & c
\end{array}
$$

admits six different canonical TRSs:

$$
\begin{array}{rcl}
f(g(b,b)) & \xrightarrow{1} & g(b,b) \\
a & \xrightarrow{2} & g(b,b) \\
h(g(b,b)) & \xrightarrow{3} & b \\
i(g(b,b)) & \xrightarrow{4} & c
\end{array}
\quad \overset{2}{\Longleftrightarrow} \quad
\begin{array}{rcl}
f(a) & \xrightarrow{1} & a \\
g(b,b) & \xrightarrow{2} & a \\
h(a) & \xrightarrow{3} & b \\
i(a) & \xrightarrow{4} & c
\end{array}
\quad \overset{3}{\Longleftrightarrow} \quad
\begin{array}{rcl}
f(a) & \xrightarrow{1} & a \\
g(h(a),h(a)) & \xrightarrow{2} & a \\
b & \xrightarrow{3} & h(a) \\
i(a) & \xrightarrow{4} & c
\end{array}
$$

$$\Big\Updownarrow 4$$

$$
\begin{array}{rcl}
f(g(b,b)) & \xrightarrow{1} & g(b,b) \\
a & \xrightarrow{2} & g(b,b) \\
h(g(b,b)) & \xrightarrow{3} & b \\
c & \xrightarrow{4} & i(g(b,b))
\end{array}
$$

Example (Snyder 1993)

$$
\begin{aligned}
f(a) &\approx g(b,b) & \qquad f(f(a)) &\approx a & \qquad f(f(f(a))) &\approx a \\
g(b,h(a)) &\approx g(b,b) & h(a) &\approx b & i(f(a)) &\approx c
\end{aligned}
$$

admits six different canonical TRSs:

$$
\begin{array}{ccccc}
\begin{aligned}
f(g(b,b)) &\xrightarrow{1} g(b,b) \\
a &\xrightarrow{2} g(b,b) \\
h(g(b,b)) &\xrightarrow{3} b \\
i(g(b,b)) &\xrightarrow{4} c
\end{aligned}
& \overset{2}{\Longleftrightarrow} &
\begin{aligned}
f(a) &\xrightarrow{1} a \\
g(b,b) &\xrightarrow{2} a \\
h(a) &\xrightarrow{3} b \\
{\color{red} i(a)} &\mathrel{\color{red}\xrightarrow{4}} {\color{red} c}
\end{aligned}
& \overset{3}{\Longleftrightarrow} &
\begin{aligned}
f(a) &\xrightarrow{1} a \\
g(h(a),h(a)) &\xrightarrow{2} a \\
b &\xrightarrow{3} h(a) \\
i(a) &\xrightarrow{4} c
\end{aligned}
\\
\Big\Updownarrow 4 & & \Big\Updownarrow 4 & & \\
\begin{aligned}
f(g(b,b)) &\xrightarrow{1} g(b,b) \\
a &\xrightarrow{2} g(b,b) \\
h(g(b,b)) &\xrightarrow{3} b \\
c &\xrightarrow{4} i(g(b,b))
\end{aligned}
& &
\begin{aligned}
f(a) &\xrightarrow{1} a \\
g(b,b) &\xrightarrow{2} a \\
h(a) &\xrightarrow{3} b \\
{\color{red} c} &\mathrel{\color{red}\xrightarrow{4}} {\color{red} i(a)}
\end{aligned}
& &
\end{array}
$$

Example (Snyder 1993)

$$\begin{array}{rclcrclcrcl}
f(a) & \approx & g(b,b) & \qquad & f(f(a)) & \approx & a & \qquad & f(f(f(a))) & \approx & a \\
g(b,h(a)) & \approx & g(b,b) & & h(a) & \approx & b & & i(f(a)) & \approx & c
\end{array}$$

admits six different canonical TRSs:

$$\begin{array}{ccccc}
\begin{array}{rcl}
f(g(b,b)) & \xrightarrow{1} & g(b,b) \\
a & \xrightarrow{2} & g(b,b) \\
h(g(b,b)) & \xrightarrow{3} & b \\
i(g(b,b)) & \xrightarrow{4} & c
\end{array}
& \overset{2}{\Longleftrightarrow} &
\begin{array}{rcl}
f(a) & \xrightarrow{1} & a \\
g(b,b) & \xrightarrow{2} & a \\
h(a) & \xrightarrow{3} & b \\
i(a) & \xrightarrow{4} & c
\end{array}
& \overset{3}{\Longleftrightarrow} &
\begin{array}{rcl}
f(a) & \xrightarrow{1} & a \\
g(h(a),h(a)) & \xrightarrow{2} & a \\
b & \xrightarrow{3} & h(a) \\
i(a) & \xrightarrow{4} & c
\end{array}
\\
\Updownarrow 4 & & \Updownarrow 4 & & \\
\begin{array}{rcl}
f(g(b,b)) & \xrightarrow{1} & g(b,b) \\
a & \xrightarrow{2} & g(b,b) \\
h(g(b,b)) & \xrightarrow{3} & b \\
c & \xrightarrow{4} & i(g(b,b))
\end{array}
& \overset{2}{\Longleftrightarrow} &
\begin{array}{rcl}
f(a) & \xrightarrow{1} & a \\
g(b,b) & \xrightarrow{2} & a \\
h(a) & \xrightarrow{3} & b \\
c & \xrightarrow{4} & i(a)
\end{array}
& &
\end{array}$$

Example (Snyder 1993)

$$\begin{array}{rcl} f(a) & \approx & g(b,b) \\ g(b,h(a)) & \approx & g(b,b) \end{array} \qquad \begin{array}{rcl} f(f(a)) & \approx & a \\ h(a) & \approx & b \end{array} \qquad \begin{array}{rcl} f(f(f(a))) & \approx & a \\ i(f(a)) & \approx & c \end{array}$$

admits six different canonical TRSs:

$$\begin{array}{ccccc}
\begin{array}{rcl} f(g(b,b)) & \xrightarrow{1} & g(b,b) \\ a & \xrightarrow{2} & g(b,b) \\ h(g(b,b)) & \xrightarrow{3} & b \\ i(g(b,b)) & \xrightarrow{4} & c \end{array}
& \overset{2}{\Longleftrightarrow} &
\begin{array}{rcl} f(a) & \xrightarrow{1} & a \\ g(b,b) & \xrightarrow{2} & a \\ h(a) & \xrightarrow{3} & b \\ i(a) & \xrightarrow{4} & c \end{array}
& \overset{3}{\Longleftrightarrow} &
\begin{array}{rcl} f(a) & \xrightarrow{1} & a \\ g(h(a),h(a)) & \xrightarrow{2} & a \\ b & \xrightarrow{3} & h(a) \\ i(a) & \xrightarrow{4} & c \end{array}
\\
\Updownarrow 4 & & \Updownarrow 4 & & \Updownarrow 4
\\
\begin{array}{rcl} f(g(b,b)) & \xrightarrow{1} & g(b,b) \\ a & \xrightarrow{2} & g(b,b) \\ h(g(b,b)) & \xrightarrow{3} & b \\ c & \xrightarrow{4} & i(g(b,b)) \end{array}
& \overset{2}{\Longleftrightarrow} &
\begin{array}{rcl} f(a) & \xrightarrow{1} & a \\ g(b,b) & \xrightarrow{2} & a \\ h(a) & \xrightarrow{3} & b \\ c & \xrightarrow{4} & i(a) \end{array}
& \overset{3}{\Longleftrightarrow} &
\begin{array}{rcl} f(a) & \xrightarrow{1} & a \\ g(h(a),h(a)) & \xrightarrow{2} & a \\ b & \xrightarrow{3} & h(a) \\ c & \xrightarrow{4} & i(a) \end{array}
\end{array}$$

Theorem (IWC 2023)

- if $\mathcal{R} \implies \mathcal{S}$ and $\mathcal{R}$ is canonical then $\mathcal{S}$ is canonical and equivalent to $\mathcal{R}$

Theorem (IWC 2023)

- if $\mathcal{R} \Longrightarrow \mathcal{S}$ and $\mathcal{R}$ is canonical then $\mathcal{S}$ is canonical and equivalent to $\mathcal{R}$
- if $\mathcal{R}$ and $\mathcal{S}$ are equivalent ground canonical TRSs then $\mathcal{R} \Longrightarrow^* \mathcal{S}$

Theorem (IWC 2023)

- if $\mathcal{R} \Longrightarrow \mathcal{S}$ and $\mathcal{R}$ is canonical then $\mathcal{S}$ is canonical and equivalent to $\mathcal{R}$
- if $\mathcal{R}$ and $\mathcal{S}$ are equivalent ground canonical TRSs then $\mathcal{R} \Longrightarrow^* \mathcal{S}$

Remark

- formalized in Isabelle/HOL by Joshua Ocker (2024)

Theorem (IWC 2023)

- if $\mathcal{R} \Longrightarrow \mathcal{S}$ and $\mathcal{R}$ is canonical then $\mathcal{S}$ is canonical and equivalent to $\mathcal{R}$
- if $\mathcal{R}$ and $\mathcal{S}$ are equivalent ground canonical TRSs then $\mathcal{R} \Longrightarrow^* \mathcal{S}$

Remark

- formalized in Isabelle/HOL by Joshua Ocker (2024)

Question

- 2^k bound on number of equivalent canonical TRSs derivable from Snyder's transformation ?

Theorem (IWC 2023)

- if $\mathcal{R} \Longrightarrow \mathcal{S}$ and $\mathcal{R}$ is canonical then $\mathcal{S}$ is canonical and equivalent to $\mathcal{R}$
- if $\mathcal{R}$ and $\mathcal{S}$ are equivalent ground canonical TRSs then $\mathcal{R} \Longrightarrow^* \mathcal{S}$

Remark

- formalized in Isabelle/HOL by Joshua Ocker (2024)

Question

- 2^k bound on number of equivalent canonical TRSs derivable from Snyder's transformation ?

Definition

finite set T of ground terms and $t \in T$

- $T_{-t} = T \setminus \{t\}$

Theorem (IWC 2023)

- if $\mathcal{R} \Longrightarrow \mathcal{S}$ and $\mathcal{R}$ is canonical then $\mathcal{S}$ is canonical and equivalent to $\mathcal{R}$
- if $\mathcal{R}$ and $\mathcal{S}$ are equivalent ground canonical TRSs then $\mathcal{R} \Longrightarrow^* \mathcal{S}$

Remark

- formalized in Isabelle/HOL by Joshua Ocker (2024)

Question

- 2^k bound on number of equivalent canonical TRSs derivable from Snyder's transformation ?

Definition

finite set T of ground terms and $t \in T$

- $T_{-t} = T \setminus \{t\}$
- $\mathcal{T}_{\to t}$ denotes TRS $\{s \to t \mid s \in T_{-t}\}$

Outline

Definition

for ground canonical TRS $\mathcal{R}$ we define set $\mathsf{E}(\mathcal{R})$ inductively

Definition

for ground canonical TRS $\mathcal{R}$ we define set $\mathsf{E}(\mathcal{R})$ inductively

- if $\mathcal{R} = \varnothing$ then $\mathsf{E}(\mathcal{R}) = \{\varnothing\}$

Definition

for ground canonical TRS $\mathcal{R}$ we define set $\mathsf{E}(\mathcal{R})$ inductively

- if $\mathcal{R} = \varnothing$ then $\mathsf{E}(\mathcal{R}) = \{\varnothing\}$
- if $\mathcal{R} \neq \varnothing$ then we choose right−hand side $t \in \mathsf{RHS}(\mathcal{R})$

Definition

for ground canonical TRS $\mathcal{R}$ we define set $\mathsf{E}(\mathcal{R})$ inductively

- if $\mathcal{R} = \varnothing$ then $\mathsf{E}(\mathcal{R}) = \{\varnothing\}$
- if $\mathcal{R} \neq \varnothing$ then we choose right–hand side $t \in \mathsf{RHS}(\mathcal{R})$ arbitrarily

Definition

for ground canonical TRS $\mathcal{R}$ we define set $\mathsf{E}(\mathcal{R})$ inductively

- if $\mathcal{R} = \varnothing$ then $\mathsf{E}(\mathcal{R}) = \{\varnothing\}$
- if $\mathcal{R} \neq \varnothing$ then we choose right–hand side $t \in \mathsf{RHS}(\mathcal{R})$ arbitrarily and let
  - $T = \{t\} \cup \{s \mid s \to t \in \mathcal{R}\}$

Definition

for ground canonical TRS $\mathcal{R}$ we define set $\mathsf{E}(\mathcal{R})$ inductively

- if $\mathcal{R} = \varnothing$ then $\mathsf{E}(\mathcal{R}) = \{\varnothing\}$
- if $\mathcal{R} \neq \varnothing$ then we choose right-hand side $t \in \mathsf{RHS}(\mathcal{R})$ arbitrarily and let
  - $T = \{t\} \cup \{s \mid s \to t \in \mathcal{R}\}$ and $\mathcal{S} = \mathcal{R} \setminus \mathcal{T}_{\to t}$

Definition

for ground canonical TRS $\mathcal{R}$ we define set $\mathsf{E}(\mathcal{R})$ inductively

- if $\mathcal{R} = \varnothing$ then $\mathsf{E}(\mathcal{R}) = \{\varnothing\}$
- if $\mathcal{R} \neq \varnothing$ then we choose right–hand side $t \in \mathsf{RHS}(\mathcal{R})$ arbitrarily and let
  - $T = \{t\} \cup \{s \mid s \to t \in \mathcal{R}\}$ and $\mathcal{S} = \mathcal{R} \setminus \mathcal{T}_{\to t}$
  - fresh constant $\bullet_t$

Definition

for ground canonical TRS $\mathcal{R}$ we define set $\mathsf{E}(\mathcal{R})$ inductively

- if $\mathcal{R} = \varnothing$ then $\mathsf{E}(\mathcal{R}) = \{\varnothing\}$
- if $\mathcal{R} \neq \varnothing$ then we choose right–hand side $t \in \mathsf{RHS}(\mathcal{R})$ arbitrarily and let
  - $T = \{t\} \cup \{s \mid s \to t \in \mathcal{R}\}$ and $\mathcal{S} = \mathcal{R} \setminus \mathcal{T}_{\to t}$
  - fresh constant $\bullet_t$
  - $\mathsf{E}(\mathcal{R})$ consists of all TRSs

$$(\mathcal{U} \cup \mathcal{V}_{\to v})\{v/\bullet_t\}$$

Definition

for ground canonical TRS $\mathcal{R}$ we define set $\mathsf{E}(\mathcal{R})$ inductively

- if $\mathcal{R} = \varnothing$ then $\mathsf{E}(\mathcal{R}) = \{\varnothing\}$
- if $\mathcal{R} \neq \varnothing$ then we choose right−hand side $t \in \mathsf{RHS}(\mathcal{R})$ arbitrarily and let
  - $T = \{t\} \cup \{s \mid s \to t \in \mathcal{R}\}$ and $\mathcal{S} = \mathcal{R} \setminus \mathcal{T}_{\to t}$
  - fresh constant $\bullet_t$
  - $\mathsf{E}(\mathcal{R})$ consists of all TRSs

$$(\mathcal{U} \cup \mathcal{V}_{\to v})\{v/\bullet_t\}$$

  such that $\mathcal{U} \in \mathsf{E}(\mathcal{S}\{\bullet_t/t\})$

Definition

for ground canonical TRS $\mathcal{R}$ we define set $\mathsf{E}(\mathcal{R})$ inductively

- if $\mathcal{R} = \varnothing$ then $\mathsf{E}(\mathcal{R}) = \{\varnothing\}$
- if $\mathcal{R} \neq \varnothing$ then we choose right–hand side $t \in \mathsf{RHS}(\mathcal{R})$ arbitrarily and let
  - $T = \{t\} \cup \{s \mid s \to t \in \mathcal{R}\}$ and $\mathcal{S} = \mathcal{R} \setminus \mathcal{T}_{\to t}$
  - fresh constant $\bullet_t$
  - $\mathsf{E}(\mathcal{R})$ consists of all TRSs

$$(\mathcal{U} \cup \mathcal{V}_{\to v})\{v/\bullet_t\}$$

such that $\mathcal{U} \in \mathsf{E}(\mathcal{S}\{\bullet_t/t\})$ and $V = (\{t\} \cup T_{-t}\{\bullet_t/t\}){\downarrow_{\mathcal{U}}}$

Definition

for ground canonical TRS $\mathcal{R}$ we define set $\mathsf{E}(\mathcal{R})$ inductively

- if $\mathcal{R} = \varnothing$ then $\mathsf{E}(\mathcal{R}) = \{\varnothing\}$
- if $\mathcal{R} \neq \varnothing$ then we choose right–hand side $t \in \mathsf{RHS}(\mathcal{R})$ arbitrarily and let
  - $T = \{t\} \cup \{s \mid s \to t \in \mathcal{R}\}$ and $\mathcal{S} = \mathcal{R} \setminus \mathcal{T}_{\to t}$
  - fresh constant $\bullet_t$
  - $\mathsf{E}(\mathcal{R})$ consists of all TRSs

$$(\mathcal{U} \cup \mathcal{V}_{\to v})\{v / \bullet_t\}$$

such that $\mathcal{U} \in \mathsf{E}(\mathcal{S}\{\bullet_t / t\})$ and $V = (\{t\} \cup T_{-t}\{\bullet_t / t\})\downarrow_{\mathcal{U}}$ and $v \in V$ such that $\bullet_t \notin v$

Example

TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}$$

Example

TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}$$

- $t = \mathsf{g}(\mathsf{b},\mathsf{b})$

Example

TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}$$

- $t = \mathsf{g}(\mathsf{b},\mathsf{b})$ and $T = \{t, \mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})), \mathsf{a}\}$

Example

TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}$$

- $t = \mathsf{g}(\mathsf{b},\mathsf{b})$ and $T = \{t, \mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})), \mathsf{a}\}$ and $\mathcal{T}_{\to t} = \{\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}), \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b})\}$

Example

TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}$$

- $t = \mathsf{g}(\mathsf{b},\mathsf{b})$ and $T = \{t, \mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})), \mathsf{a}\}$ and $\mathcal{T}_{\to t} = \{\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}), \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b})\}$
- $\mathcal{S} = \{\mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}\}$

Example

TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}$$

- $t = \mathsf{g}(\mathsf{b},\mathsf{b})$ and $T = \{t, \mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})), \mathsf{a}\}$ and $\mathcal{T}_{\to t} = \{\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}), \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b})\}$
- $\mathcal{S} = \{\mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}\}$ and $\mathcal{S}\{\bullet_t/t\} = \{\mathsf{h}(\bullet_t) \to \mathsf{b}\}$

Example

TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}$$

- $t = \mathsf{g}(\mathsf{b},\mathsf{b})$ and $T = \{t, \mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})), \mathsf{a}\}$ and $\mathcal{T}_{\to t} = \{\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}), \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b})\}$
- $\mathcal{S} = \{\mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}\}$ and $\mathcal{S}\{\bullet_t/t\} = \{\mathsf{h}(\bullet_t) \to \mathsf{b}\}$
- $\mathsf{E}(\mathcal{S}\{\bullet_t/t\}) = \{\mathcal{U}_1, \mathcal{U}_2\}$ with $\mathcal{U}_1 = \{\mathsf{h}(\bullet_t) \to \mathsf{b}\}$ and $\mathcal{U}_2 = \{\mathsf{b} \to \mathsf{h}(\bullet_t)\}$

Example

TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}$$

- $t = \mathsf{g}(\mathsf{b},\mathsf{b})$ and $T = \{t, \mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})), \mathsf{a}\}$ and $\mathcal{T}_{\to t} = \{\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}), \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b})\}$
- $\mathcal{S} = \{\mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}\}$ and $\mathcal{S}\{\bullet_t/t\} = \{\mathsf{h}(\bullet_t) \to \mathsf{b}\}$
- $\mathsf{E}(\mathcal{S}\{\bullet_t/t\}) = \{\mathcal{U}_1, \mathcal{U}_2\}$ with $\mathcal{U}_1 = \{\mathsf{h}(\bullet_t) \to \mathsf{b}\}$ and $\mathcal{U}_2 = \{\mathsf{b} \to \mathsf{h}(\bullet_t)\}$

1. $V = \{t, \mathsf{f}(\bullet_t), \mathsf{a}\}{\downarrow_{\mathcal{U}_1}} = \{\mathsf{g}(\mathsf{b},\mathsf{b}), \mathsf{f}(\bullet_t), \mathsf{a}\}$

Example

TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}$$

- $t = \mathsf{g}(\mathsf{b},\mathsf{b})$ and $T = \{t, \mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})), \mathsf{a}\}$ and $\mathcal{T}_{\to t} = \{\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}), \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b})\}$
- $\mathcal{S} = \{\mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}\}$ and $\mathcal{S}\{\bullet_t/t\} = \{\mathsf{h}(\bullet_t) \to \mathsf{b}\}$
- $\mathsf{E}(\mathcal{S}\{\bullet_t/t\}) = \{\mathcal{U}_1, \mathcal{U}_2\}$ with $\mathcal{U}_1 = \{\mathsf{h}(\bullet_t) \to \mathsf{b}\}$ and $\mathcal{U}_2 = \{\mathsf{b} \to \mathsf{h}(\bullet_t)\}$

1. $V = \{t, \mathsf{f}(\bullet_t), \mathsf{a}\}{\downarrow_{\mathcal{U}_1}} = \{\mathsf{g}(\mathsf{b},\mathsf{b}), \mathsf{f}(\bullet_t), \mathsf{a}\} \quad \Longrightarrow \quad v = \mathsf{g}(\mathsf{b},\mathsf{b})$

$$\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}$$

Example

TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}$$

- $t = \mathsf{g}(\mathsf{b},\mathsf{b})$ and $T = \{t, \mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})), \mathsf{a}\}$ and $\mathcal{T}_{\to t} = \{\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}), \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b})\}$
- $\mathcal{S} = \{\mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}\}$ and $\mathcal{S}\{\bullet_t/t\} = \{\mathsf{h}(\bullet_t) \to \mathsf{b}\}$
- $\mathsf{E}(\mathcal{S}\{\bullet_t/t\}) = \{\mathcal{U}_1, \mathcal{U}_2\}$ with $\mathcal{U}_1 = \{\mathsf{h}(\bullet_t) \to \mathsf{b}\}$ and $\mathcal{U}_2 = \{\mathsf{b} \to \mathsf{h}(\bullet_t)\}$

1. $V = \{t, \mathsf{f}(\bullet_t), \mathsf{a}\}{\downarrow_{\mathcal{U}_1}} = \{\mathsf{g}(\mathsf{b},\mathsf{b}), \mathsf{f}(\bullet_t), \mathsf{a}\} \implies v = \mathsf{g}(\mathsf{b},\mathsf{b})$ or $v = \mathsf{a}$

$$\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}$$

$$\mathsf{f}(\mathsf{a}) \to \mathsf{a} \qquad \mathsf{g}(\mathsf{b},\mathsf{b}) \to \mathsf{a} \qquad \mathsf{h}(\mathsf{a}) \to \mathsf{b}$$

Example

TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}$$

- $t = \mathsf{g}(\mathsf{b},\mathsf{b})$ and $T = \{t, \mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})), \mathsf{a}\}$ and $\mathcal{T}_{\to t} = \{\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}), \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b})\}$
- $\mathcal{S} = \{\mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}\}$ and $\mathcal{S}\{\bullet_t/t\} = \{\mathsf{h}(\bullet_t) \to \mathsf{b}\}$
- $\mathsf{E}(\mathcal{S}\{\bullet_t/t\}) = \{\mathcal{U}_1, \mathcal{U}_2\}$ with $\mathcal{U}_1 = \{\mathsf{h}(\bullet_t) \to \mathsf{b}\}$ and $\mathcal{U}_2 = \{\mathsf{b} \to \mathsf{h}(\bullet_t)\}$

1. $V = \{t, \mathsf{f}(\bullet_t), \mathsf{a}\}{\downarrow_{\mathcal{U}_1}} = \{\mathsf{g}(\mathsf{b},\mathsf{b}), \mathsf{f}(\bullet_t), \mathsf{a}\} \implies v = \mathsf{g}(\mathsf{b},\mathsf{b})$ or $v = \mathsf{a}$

$$\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}$$

$$\mathsf{f}(\mathsf{a}) \to \mathsf{a} \qquad \mathsf{g}(\mathsf{b},\mathsf{b}) \to \mathsf{a} \qquad \mathsf{h}(\mathsf{a}) \to \mathsf{b}$$

2. $V = \{t, \mathsf{f}(\bullet_t), \mathsf{a}\}{\downarrow_{\mathcal{U}_2}} = \{\mathsf{g}(\mathsf{h}(\bullet_t),\mathsf{h}(\bullet_t)), \mathsf{f}(\bullet_t), \mathsf{a}\}$

Example

TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}$$

- $t = \mathsf{g}(\mathsf{b},\mathsf{b})$ and $T = \{t, \mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})), \mathsf{a}\}$ and $\mathcal{T}_{\to t} = \{\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}), \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b})\}$
- $\mathcal{S} = \{\mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}\}$ and $\mathcal{S}\{\bullet_t/t\} = \{\mathsf{h}(\bullet_t) \to \mathsf{b}\}$
- $\mathsf{E}(\mathcal{S}\{\bullet_t/t\}) = \{\mathcal{U}_1, \mathcal{U}_2\}$ with $\mathcal{U}_1 = \{\mathsf{h}(\bullet_t) \to \mathsf{b}\}$ and $\mathcal{U}_2 = \{\mathsf{b} \to \mathsf{h}(\bullet_t)\}$

1. $V = \{t, \mathsf{f}(\bullet_t), \mathsf{a}\}{\downarrow_{\mathcal{U}_1}} = \{\mathsf{g}(\mathsf{b},\mathsf{b}), \mathsf{f}(\bullet_t), \mathsf{a}\} \quad \Longrightarrow \quad v = \mathsf{g}(\mathsf{b},\mathsf{b})$ or $v = \mathsf{a}$

$$\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}$$

$$\mathsf{f}(\mathsf{a}) \to \mathsf{a} \qquad \mathsf{g}(\mathsf{b},\mathsf{b}) \to \mathsf{a} \qquad \mathsf{h}(\mathsf{a}) \to \mathsf{b}$$

2. $V = \{t, \mathsf{f}(\bullet_t), \mathsf{a}\}{\downarrow_{\mathcal{U}_2}} = \{\mathsf{g}(\mathsf{h}(\bullet_t),\mathsf{h}(\bullet_t)), \mathsf{f}(\bullet_t), \mathsf{a}\} \quad \Longrightarrow \quad v = \mathsf{a}$

$$\mathsf{f}(\mathsf{a}) \to \mathsf{a} \qquad \mathsf{g}(\mathsf{h}(\mathsf{a}),\mathsf{h}(\mathsf{a})) \to \mathsf{a} \qquad \mathsf{b} \to \mathsf{h}(\mathsf{a})$$

Example

TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}$$

- $t = \mathsf{b}$ and $T = \{t, \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b}))\}$ and $\mathcal{T}_{\to t} = \{\mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}\}$

Example

TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}$$

- $t = \mathsf{b}$ and $T = \{t, \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b}))\}$ and $\mathcal{T}_{\to t} = \{\mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}\}$
- $\mathcal{S}\{\bullet_t/t\} = \{\mathsf{f}(\mathsf{g}(\bullet_t,\bullet_t)) \to \mathsf{g}(\bullet_t,\bullet_t),\ \mathsf{a} \to \mathsf{g}(\bullet_t,\bullet_t)\}$

Example

TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}$$

- $t = \mathsf{b}$ and $T = \{t, \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b}))\}$ and $\mathcal{T}_{\to t} = \{\mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}\}$
- $\mathcal{S}\{\bullet_t/t\} = \{\mathsf{f}(\mathsf{g}(\bullet_t,\bullet_t)) \to \mathsf{g}(\bullet_t,\bullet_t), \mathsf{a} \to \mathsf{g}(\bullet_t,\bullet_t)\}$
- $\mathsf{E}(\mathcal{S}\{\bullet_t/t\}) = \{\mathcal{U}_1, \mathcal{U}_2\}$ with $\mathcal{U}_1 = \mathcal{S}\{\bullet_t/t\}$ and $\mathcal{U}_2 = \{\mathsf{f}(\mathsf{a}) \to \mathsf{a}, \mathsf{g}(\bullet_t,\bullet_t) \to \mathsf{a}\}$

Example

TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}$$

- $t = \mathsf{b}$ and $T = \{t, \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b}))\}$ and $\mathcal{T}_{\to t} = \{\mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}\}$
- $\mathcal{S}\{\bullet_t/t\} = \{\mathsf{f}(\mathsf{g}(\bullet_t,\bullet_t)) \to \mathsf{g}(\bullet_t,\bullet_t), \mathsf{a} \to \mathsf{g}(\bullet_t,\bullet_t)\}$
- $\mathsf{E}(\mathcal{S}\{\bullet_t/t\}) = \{\mathcal{U}_1, \mathcal{U}_2\}$ with $\mathcal{U}_1 = \mathcal{S}\{\bullet_t/t\}$ and $\mathcal{U}_2 = \{\mathsf{f}(\mathsf{a}) \to \mathsf{a}, \mathsf{g}(\bullet_t,\bullet_t) \to \mathsf{a}\}$

1. $V = \{t, \mathsf{h}(\mathsf{g}(\bullet_t,\bullet_t))\}{\downarrow_{\mathcal{U}_1}} = \{\mathsf{b}, \mathsf{h}(\mathsf{g}(\bullet_t,\bullet_t))\}$

Example

TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}$$

- $t = \mathsf{b}$ and $T = \{t, \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b}))\}$ and $\mathcal{T}_{\to t} = \{\mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}\}$
- $\mathcal{S}\{\bullet_t/t\} = \{\mathsf{f}(\mathsf{g}(\bullet_t,\bullet_t)) \to \mathsf{g}(\bullet_t,\bullet_t),\ \mathsf{a} \to \mathsf{g}(\bullet_t,\bullet_t)\}$
- $\mathsf{E}(\mathcal{S}\{\bullet_t/t\}) = \{\mathcal{U}_1, \mathcal{U}_2\}$ with $\mathcal{U}_1 = \mathcal{S}\{\bullet_t/t\}$ and $\mathcal{U}_2 = \{\mathsf{f}(\mathsf{a}) \to \mathsf{a},\ \mathsf{g}(\bullet_t,\bullet_t) \to \mathsf{a}\}$

1. $V = \{t, \mathsf{h}(\mathsf{g}(\bullet_t,\bullet_t))\}{\downarrow_{\mathcal{U}_1}} = \{\mathsf{b}, \mathsf{h}(\mathsf{g}(\bullet_t,\bullet_t))\} \implies v = \mathsf{b}$

$$\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}$$

Example

TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}$$

- $t = \mathsf{b}$ and $T = \{t, \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b}))\}$ and $\mathcal{T}_{\to t} = \{\mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}\}$
- $\mathcal{S}\{\bullet_t/t\} = \{\mathsf{f}(\mathsf{g}(\bullet_t,\bullet_t)) \to \mathsf{g}(\bullet_t,\bullet_t),\ \mathsf{a} \to \mathsf{g}(\bullet_t,\bullet_t)\}$
- $\mathrm{E}(\mathcal{S}\{\bullet_t/t\}) = \{\mathcal{U}_1, \mathcal{U}_2\}$ with $\mathcal{U}_1 = \mathcal{S}\{\bullet_t/t\}$ and $\mathcal{U}_2 = \{\mathsf{f}(\mathsf{a}) \to \mathsf{a},\ \mathsf{g}(\bullet_t,\bullet_t) \to \mathsf{a}\}$

1. $V = \{t, \mathsf{h}(\mathsf{g}(\bullet_t,\bullet_t))\}{\downarrow_{\mathcal{U}_1}} = \{\mathsf{b}, \mathsf{h}(\mathsf{g}(\bullet_t,\bullet_t))\} \quad \Longrightarrow \quad v = \mathsf{b}$

$$\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}$$

2. $V = \{t, \mathsf{h}(\mathsf{g}(\bullet_t,\bullet_t))\}{\downarrow_{\mathcal{U}_2}} = \{\mathsf{b}, \mathsf{h}(\mathsf{a})\}$

Example

TRS $\mathcal{R}$

$$\mathsf{f(g(b,b))} \to \mathsf{g(b,b)} \qquad \mathsf{a} \to \mathsf{g(b,b)} \qquad \mathsf{h(g(b,b))} \to \mathsf{b}$$

- $t = \mathsf{b}$ and $T = \{t, \mathsf{h(g(b,b))}\}$ and $\mathcal{T}_{\to t} = \{\mathsf{h(g(b,b))} \to \mathsf{b}\}$
- $\mathcal{S}\{\bullet_t/t\} = \{\mathsf{f}(\mathsf{g}(\bullet_t, \bullet_t)) \to \mathsf{g}(\bullet_t, \bullet_t), \mathsf{a} \to \mathsf{g}(\bullet_t, \bullet_t)\}$
- $\mathsf{E}(\mathcal{S}\{\bullet_t/t\}) = \{\mathcal{U}_1, \mathcal{U}_2\}$ with $\mathcal{U}_1 = \mathcal{S}\{\bullet_t/t\}$ and $\mathcal{U}_2 = \{\mathsf{f(a)} \to \mathsf{a}, \mathsf{g}(\bullet_t, \bullet_t) \to \mathsf{a}\}$

1. $V = \{t, \mathsf{h}(\mathsf{g}(\bullet_t, \bullet_t))\}{\downarrow_{\mathcal{U}_1}} = \{\mathsf{b}, \mathsf{h}(\mathsf{g}(\bullet_t, \bullet_t))\} \implies v = \mathsf{b}$

$$\mathsf{f(g(b,b))} \to \mathsf{g(b,b)} \qquad \mathsf{a} \to \mathsf{g(b,b)} \qquad \mathsf{h(g(b,b))} \to \mathsf{b}$$

2. $V = \{t, \mathsf{h}(\mathsf{g}(\bullet_t, \bullet_t))\}{\downarrow_{\mathcal{U}_2}} = \{\mathsf{b}, \mathsf{h(a)}\} \implies v = \mathsf{b}$

$$\mathsf{f(a)} \to \mathsf{a} \qquad \mathsf{g(b,b)} \to \mathsf{a} \qquad \mathsf{h(a)} \to \mathsf{b}$$

Example

TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}$$

- $t = \mathsf{b}$ and $T = \{t, \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b}))\}$ and $\mathcal{T}_{\to t} = \{\mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}\}$
- $\mathcal{S}\{\bullet_t/t\} = \{\mathsf{f}(\mathsf{g}(\bullet_t,\bullet_t)) \to \mathsf{g}(\bullet_t,\bullet_t),\ \mathsf{a} \to \mathsf{g}(\bullet_t,\bullet_t)\}$
- $\mathsf{E}(\mathcal{S}\{\bullet_t/t\}) = \{\mathcal{U}_1, \mathcal{U}_2\}$ with $\mathcal{U}_1 = \mathcal{S}\{\bullet_t/t\}$ and $\mathcal{U}_2 = \{\mathsf{f}(\mathsf{a}) \to \mathsf{a},\ \mathsf{g}(\bullet_t,\bullet_t) \to \mathsf{a}\}$

1. $V = \{t, \mathsf{h}(\mathsf{g}(\bullet_t,\bullet_t))\}\downarrow_{\mathcal{U}_1} = \{\mathsf{b}, \mathsf{h}(\mathsf{g}(\bullet_t,\bullet_t))\} \implies v = \mathsf{b}$

$$\mathsf{f}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{a} \to \mathsf{g}(\mathsf{b},\mathsf{b}) \qquad \mathsf{h}(\mathsf{g}(\mathsf{b},\mathsf{b})) \to \mathsf{b}$$

2. $V = \{t, \mathsf{h}(\mathsf{g}(\bullet_t,\bullet_t))\}\downarrow_{\mathcal{U}_2} = \{\mathsf{b}, \mathsf{h}(\mathsf{a})\} \implies v = \mathsf{b}$ or $v = \mathsf{h}(\mathsf{a})$

$$\mathsf{f}(\mathsf{a}) \to \mathsf{a} \qquad \mathsf{g}(\mathsf{b},\mathsf{b}) \to \mathsf{a} \qquad \mathsf{h}(\mathsf{a}) \to \mathsf{b}$$

$$\mathsf{f}(\mathsf{a}) \to \mathsf{a} \qquad \mathsf{g}(\mathsf{h}(\mathsf{a}),\mathsf{h}(\mathsf{a})) \to \mathsf{a} \qquad \mathsf{b} \to \mathsf{h}(\mathsf{a})$$

Definition

$\mathsf{P}(\mathcal{R}, T, \bullet)$ denotes conjunction of following conditions:

① $\mathcal{R}$ is canonical ground TRS

Definition

$\mathsf{P}(\mathcal{R}, T, \bullet)$ denotes conjunction of following conditions:

① $\mathcal{R}$ is canonical ground TRS

② $T \subseteq \mathsf{NF}(\mathcal{R})$ is finite set of ground terms

Definition

$\mathsf{P}(\mathcal{R}, T, \bullet)$ denotes conjunction of following conditions:

① $\mathcal{R}$ is canonical ground TRS

② $T \subseteq \mathsf{NF}(\mathcal{R})$ is finite set of ground terms

③ no $t \in T$ is subterm of term in $\mathsf{LHS}(\mathcal{R}) \cup \mathsf{RHS}(\mathcal{R}) \cup T_{-t}$

Definition

$\mathsf{P}(\mathcal{R}, T, \bullet)$ denotes conjunction of following conditions:

① $\mathcal{R}$ is canonical ground TRS

② $T \subseteq \mathsf{NF}(\mathcal{R})$ is finite set of ground terms

③ no $t \in T$ is subterm of term in $\mathsf{LHS}(\mathcal{R}) \cup \mathsf{RHS}(\mathcal{R}) \cup T_{-t}$

④ $\bullet \notin \mathsf{LHS}(\mathcal{R}) \cup \mathsf{RHS}(\mathcal{R}) \cup T$ is constant

Definition

$\mathsf{P}(\mathcal{R}, T, \bullet)$ denotes conjunction of following conditions:

① $\mathcal{R}$ is canonical ground TRS

② $T \subseteq \mathsf{NF}(\mathcal{R})$ is finite set of ground terms

③ no $t \in T$ is subterm of term in $\mathsf{LHS}(\mathcal{R}) \cup \mathsf{RHS}(\mathcal{R}) \cup T_{-t}$

④ $\bullet \notin \mathsf{LHS}(\mathcal{R}) \cup \mathsf{RHS}(\mathcal{R}) \cup T$ is constant

Lemma

$\mathsf{P}(\mathcal{R}, T, \bullet)$ and $\bullet \notin t \in T \implies \mathcal{R} \cup \mathcal{T}_{\to t}$ and $(\mathcal{R} \cup \mathcal{T}_{\to t})\{t/\bullet\}$ are canonical

Definition

$\mathsf{P}(\mathcal{R}, T, \bullet)$ denotes conjunction of following conditions:

① $\mathcal{R}$ is canonical ground TRS

② $T \subseteq \mathsf{NF}(\mathcal{R})$ is finite set of ground terms

③ no $t \in T$ is subterm of term in $\mathsf{LHS}(\mathcal{R}) \cup \mathsf{RHS}(\mathcal{R}) \cup T_{-t}$

④ $\bullet \notin \mathsf{LHS}(\mathcal{R}) \cup \mathsf{RHS}(\mathcal{R}) \cup T$ is constant

Lemma

$\mathsf{P}(\mathcal{R}, T, \bullet)$ and $\bullet \notin t \in T \quad \Longrightarrow \quad \mathcal{R} \cup \mathcal{T}_{\to t}$ and $(\mathcal{R} \cup \mathcal{T}_{\to t})\{t/\bullet\}$ are canonical

Lemma

$\mathcal{R} \Longrightarrow^* \mathcal{S}$ and $\mathsf{P}(\mathcal{R}, T, \bullet) \quad \Longrightarrow \quad \mathsf{P}(\mathcal{S}, T{\downarrow_{\mathcal{S}}}, \bullet)$

Lemma

$\mathcal{R} \Longrightarrow^* \mathcal{S}$ and $\bullet \notin t \in T$ and $\bullet \notin u \in U = T{\downarrow_\mathcal{S}} \quad \Longrightarrow$

TRSs $(\mathcal{R} \cup \mathcal{T}_{\to t})\{t/\bullet\}$ and $(\mathcal{S} \cup \mathcal{U}_{\to u})\{u/\bullet\}$ are equivalent

Lemma

$\mathcal{R} \Longrightarrow^* \mathcal{S}$ and $\bullet \notin t \in T$ and $\bullet \notin u \in U = T{\downarrow_{\mathcal{S}}} \quad \Longrightarrow$
TRSs $(\mathcal{R} \cup \mathcal{T}_{\to t})\{t/\bullet\}$ and $(\mathcal{S} \cup \mathcal{U}_{\to u})\{u/\bullet\}$ are equivalent

Lemma

$\mathcal{R} = \mathcal{S} \uplus \mathcal{T}_{\to t} \Longrightarrow^* \mathcal{R}'$ with $\bullet_t \notin \mathcal{R}$ and $\mathrm{P}(\mathcal{S}\{\bullet_t/t\}, \{t\} \cup T_{-t}\{\bullet_t/t\}, \bullet_t)$

Lemma

$\mathcal{R} \Longrightarrow^* \mathcal{S}$ and $\bullet \notin t \in T$ and $\bullet \notin u \in U = T{\downarrow_{\mathcal{S}}} \quad \Longrightarrow$

TRSs $(\mathcal{R} \cup \mathcal{T}_{\to t})\{t/\bullet\}$ and $(\mathcal{S} \cup \mathcal{U}_{\to u})\{u/\bullet\}$ are equivalent

Lemma

$\mathcal{R} = \mathcal{S} \uplus \mathcal{T}_{\to t} \Longrightarrow^* \mathcal{R}'$ with $\bullet_t \notin \mathcal{R}$ and $\mathrm{P}(\mathcal{S}\{\bullet_t/t\}, \{t\} \cup T_{-t}\{\bullet_t/t\}, \bullet_t) \quad \Longrightarrow$

there exist TRS $\mathcal{U}$ and term v such that

- $\mathcal{S}\{\bullet_t/t\} \Longrightarrow^* \mathcal{U}$

Lemma

$\mathcal{R} \Longrightarrow^* \mathcal{S}$ and $\bullet \notin t \in T$ and $\bullet \notin u \in U = T{\downarrow_{\mathcal{S}}} \quad \Longrightarrow$

TRSs $(\mathcal{R} \cup \mathcal{T}_{\to t})\{t/\bullet\}$ and $(\mathcal{S} \cup \mathcal{U}_{\to u})\{u/\bullet\}$ are equivalent

Lemma

$\mathcal{R} = \mathcal{S} \uplus \mathcal{T}_{\to t} \Longrightarrow^* \mathcal{R}'$ with $\bullet_t \notin \mathcal{R}$ and $\mathrm{P}(\mathcal{S}\{\bullet_t/t\}, \{t\} \cup T_{-t}\{\bullet_t/t\}, \bullet_t) \quad \Longrightarrow$

there exist TRS $\mathcal{U}$ and term v such that

- $\mathcal{S}\{\bullet_t/t\} \Longrightarrow^* \mathcal{U}$
- $\mathcal{R}' = (\mathcal{U} \cup \mathcal{V}_{\to v})\{v/\bullet_t\}$ with $V = (\{t\} \cup T_{-t}\{\bullet_t/t\}){\downarrow_{\mathcal{U}}}$ and $\bullet_t \notin v \in V$

Lemma

$\mathcal{R} \Longrightarrow^* \mathcal{S}$ and $\bullet \notin t \in T$ and $\bullet \notin u \in U = T{\downarrow_{\mathcal{S}}} \quad \Longrightarrow$

TRSs $(\mathcal{R} \cup \mathcal{T}_{\to t})\{t/\bullet\}$ and $(\mathcal{S} \cup \mathcal{U}_{\to u})\{u/\bullet\}$ are equivalent

Lemma

$\mathcal{R} = \mathcal{S} \uplus \mathcal{T}_{\to t} \Longrightarrow^* \mathcal{R}'$ with $\bullet_t \notin \mathcal{R}$ and $\mathsf{P}(\mathcal{S}\{\bullet_t/t\}, \{t\} \cup T_{-t}\{\bullet_t/t\}, \bullet_t) \quad \Longrightarrow$

there exist TRS $\mathcal{U}$ and term v such that

- $\mathcal{S}\{\bullet_t/t\} \Longrightarrow^* \mathcal{U}$
- $\mathcal{R}' = (\mathcal{U} \cup \mathcal{V}_{\to v})\{v/\bullet_t\}$ with $V = (\{t\} \cup T_{-t}\{\bullet_t/t\}){\downarrow_{\mathcal{U}}}$ and $\bullet_t \notin v \in V$
- $\mathsf{P}(\mathcal{U}, V, \bullet_t)$

Definitions

- $\mathsf{CaP}(\mathcal{R})$ denotes set of canonical presentations of given ground TRS $\mathcal{R}$

Definitions

- $\mathsf{CaP}(\mathcal{R})$ denotes set of canonical presentations of given ground TRS $\mathcal{R}$

Theorem

$\mathsf{CaP}(\mathcal{R}) = \mathsf{E}(\mathcal{R})$ for every ground canonical TRS $\mathcal{R}$

Definitions

- $\mathsf{CaP}(\mathcal{R})$ denotes set of canonical presentations of given ground TRS $\mathcal{R}$
- bound$(\mathcal{R})$ abbreviates

$$\prod_{v \in \mathsf{RHS}(\mathcal{R})} \left(1 + \#\{\ell \to r \in \mathcal{R} \mid r = v\}\right)$$

Theorem

$\mathsf{CaP}(\mathcal{R}) = \mathsf{E}(\mathcal{R})$ for every ground canonical TRS $\mathcal{R}$

Definitions

- $\mathsf{CaP}(\mathcal{R})$ denotes set of canonical presentations of given ground TRS $\mathcal{R}$
- $\mathsf{bound}(\mathcal{R})$ abbreviates

$$\prod_{v \in \mathsf{RHS}(\mathcal{R})} \big(1 + \#\{\ell \to r \in \mathcal{R} \mid r = v\}\big)$$

Theorem

$\mathsf{CaP}(\mathcal{R}) = \mathsf{E}(\mathcal{R})$ for every ground canonical TRS $\mathcal{R}$

Corollary

$\#\mathsf{CaP}(\mathcal{R}) \leqslant \mathsf{bound}(\mathcal{R})$ for every ground canonical TRS $\mathcal{R}$

Example (Snyder 1993)

$$\begin{aligned} f(a) &\approx g(b,b) & f(f(a)) &\approx a & f(f(f(a))) &\approx a \\ g(b,h(a)) &\approx g(b,b) & h(a) &\approx b & i(f(a)) &\approx c \end{aligned}$$

admits six different canonical TRSs:

$$\begin{array}{rcl} f(g(b,b)) &\xrightarrow{1}& g(b,b) \\ a &\xrightarrow{2}& g(b,b) \\ h(g(b,b)) &\xrightarrow{3}& b \\ i(g(b,b)) &\xrightarrow{4}& c \end{array} \quad \stackrel{2}{\Longleftrightarrow} \quad \begin{array}{rcl} f(a) &\xrightarrow{1}& a \\ g(b,b) &\xrightarrow{2}& a \\ h(a) &\xrightarrow{3}& b \\ i(a) &\xrightarrow{4}& c \end{array} \quad \stackrel{3}{\Longleftrightarrow} \quad \begin{array}{rcl} f(a) &\xrightarrow{1}& a \\ g(h(a),h(a)) &\xrightarrow{2}& a \\ b &\xrightarrow{3}& h(a) \\ i(a) &\xrightarrow{4}& c \end{array}$$

$$\Updownarrow 4 \qquad\qquad \Updownarrow 4 \qquad\qquad \Updownarrow 4$$

$$\begin{array}{rcl} f(g(b,b)) &\xrightarrow{1}& g(b,b) \\ a &\xrightarrow{2}& g(b,b) \\ h(g(b,b)) &\xrightarrow{3}& b \\ c &\xrightarrow{4}& i(g(b,b)) \end{array} \quad \stackrel{2}{\Longleftrightarrow} \quad \begin{array}{rcl} f(a) &\xrightarrow{1}& a \\ g(b,b) &\xrightarrow{2}& a \\ h(a) &\xrightarrow{3}& b \\ c &\xrightarrow{4}& i(a) \end{array} \quad \stackrel{3}{\Longleftrightarrow} \quad \begin{array}{rcl} f(a) &\xrightarrow{1}& a \\ g(h(a),h(a)) &\xrightarrow{2}& a \\ b &\xrightarrow{3}& h(a) \\ c &\xrightarrow{4}& i(a) \end{array}$$

Example

TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{a}) \to \mathsf{a} \qquad \mathsf{g}(\mathsf{b},\mathsf{b}) \to \mathsf{a} \qquad \mathsf{h}(\mathsf{a}) \to \mathsf{b} \qquad \mathsf{c} \to \mathsf{i}(\mathsf{a})$$

Example

TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{a}) \to \mathsf{a} \qquad \mathsf{g}(\mathsf{b},\mathsf{b}) \to \mathsf{a} \qquad \mathsf{h}(\mathsf{a}) \to \mathsf{b} \qquad \mathsf{c} \to \mathsf{i}(\mathsf{a})$$

- $\mathsf{CaP}(\mathcal{R}) = 6$

Example

TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{a}) \to \mathsf{a} \qquad \mathsf{g}(\mathsf{b},\mathsf{b}) \to \mathsf{a} \qquad \mathsf{h}(\mathsf{a}) \to \mathsf{b} \qquad \mathsf{c} \to \mathsf{i}(\mathsf{a})$$

- $\mathsf{CaP}(\mathcal{R}) = 6$
- $\mathsf{bound}(\mathcal{R}) = 3 \times 2 \times 2 = 12 < 16$

$$\mathsf{bound}(\mathcal{R}) = \prod_{v \in \mathsf{RHS}(\mathcal{R})} \big(1 + \#\{\ell \to r \in \mathcal{R} \mid r = v\}\big)$$

Corollary

$\#\,\mathsf{CaP}(\mathcal{R}) \leqslant \mathsf{bound}(\mathcal{R})$ for every ground canonical TRS $\mathcal{R}$

Proof

we prove $\#\,\mathsf{E}(\mathcal{R}) \leqslant \mathsf{bound}(\mathcal{R})$ by induction on $\#\,\mathcal{R}$

$$\mathsf{bound}(\mathcal{R}) = \prod_{v \in \mathsf{RHS}(\mathcal{R})} \big(1 + \#\{\ell \to r \in \mathcal{R} \mid r = v\}\big)$$

Corollary

$\#\,\mathsf{CaP}(\mathcal{R}) \leqslant \mathsf{bound}(\mathcal{R})$ for every ground canonical TRS $\mathcal{R}$

Proof

we prove $\#\,\mathsf{E}(\mathcal{R}) \leqslant \mathsf{bound}(\mathcal{R})$ by induction on $\#\,\mathcal{R}$

- $\mathcal{R} = \varnothing \quad \Longrightarrow \quad \#\,\mathsf{E}(\mathcal{R}) = \#\{\varnothing\}$

$$\mathrm{bound}(\mathcal{R}) = \prod_{v \in \mathrm{RHS}(\mathcal{R})} \big(1 + \#\{\ell \to r \in \mathcal{R} \mid r = v\}\big)$$

Corollary

$\#\,\mathrm{CaP}(\mathcal{R}) \leqslant \mathrm{bound}(\mathcal{R})$ for every ground canonical TRS $\mathcal{R}$

Proof

we prove $\#\,\mathrm{E}(\mathcal{R}) \leqslant \mathrm{bound}(\mathcal{R})$ by induction on $\#\,\mathcal{R}$

- $\mathcal{R} = \varnothing \quad \Longrightarrow \quad \#\,\mathrm{E}(\mathcal{R}) = \#\{\varnothing\} = 1 = \mathrm{bound}(\mathcal{R})$

$$\mathsf{bound}(\mathcal{R}) = \prod_{v \in \mathsf{RHS}(\mathcal{R})} \left(1 + \#\{\ell \to r \in \mathcal{R} \mid r = v\}\right)$$

Corollary

$\#\,\mathsf{CaP}(\mathcal{R}) \leqslant \mathsf{bound}(\mathcal{R})$ for every ground canonical TRS $\mathcal{R}$

Proof

we prove $\#\,\mathsf{E}(\mathcal{R}) \leqslant \mathsf{bound}(\mathcal{R})$ by induction on $\#\,\mathcal{R}$

- $\mathcal{R} = \varnothing \quad \Longrightarrow \quad \#\,\mathsf{E}(\mathcal{R}) = \#\{\varnothing\} = 1 = \mathsf{bound}(\mathcal{R})$
- suppose $\mathcal{R} \neq \varnothing$

$$\mathrm{bound}(\mathcal{R}) = \prod_{v \in \mathrm{RHS}(\mathcal{R})} \left(1 + \#\{\ell \to r \in \mathcal{R} \mid r = v\}\right)$$

Corollary

$\#\,\mathrm{CaP}(\mathcal{R}) \leqslant \mathrm{bound}(\mathcal{R})$ for every ground canonical TRS $\mathcal{R}$

Proof

we prove $\#\,\mathrm{E}(\mathcal{R}) \leqslant \mathrm{bound}(\mathcal{R})$ by induction on $\#\,\mathcal{R}$

- $\mathcal{R} = \varnothing \quad \Longrightarrow \quad \#\,\mathrm{E}(\mathcal{R}) = \#\{\varnothing\} = 1 = \mathrm{bound}(\mathcal{R})$
- suppose $\mathcal{R} \neq \varnothing$ and let $t \in \mathrm{RHS}(\mathcal{R})$

$$\mathrm{bound}(\mathcal{R}) = \prod_{v \in \mathrm{RHS}(\mathcal{R})} \big(1 + \#\{\ell \to r \in \mathcal{R} \mid r = v\}\big)$$

Corollary

$\#\,\mathrm{CaP}(\mathcal{R}) \leqslant \mathrm{bound}(\mathcal{R})$ for every ground canonical TRS $\mathcal{R}$

Proof

we prove $\#\,\mathrm{E}(\mathcal{R}) \leqslant \mathrm{bound}(\mathcal{R})$ by induction on $\#\,\mathcal{R}$

- $\mathcal{R} = \varnothing \quad \Longrightarrow \quad \#\,\mathrm{E}(\mathcal{R}) = \#\{\varnothing\} = 1 = \mathrm{bound}(\mathcal{R})$
- suppose $\mathcal{R} \neq \varnothing$ and let $t \in \mathrm{RHS}(\mathcal{R})$
- $\mathcal{R} = \mathcal{S} \uplus \mathcal{T}_{\to t}$ where $T = \{t\} \cup \{\ell \mid \ell \to t \in \mathcal{R}\}$

$$\mathsf{bound}(\mathcal{R}) = \prod_{v \in \mathsf{RHS}(\mathcal{R})} \big(1 + \#\{\ell \to r \in \mathcal{R} \mid r = v\}\big)$$

Corollary

$\#\,\mathsf{CaP}(\mathcal{R}) \leqslant \mathsf{bound}(\mathcal{R})$ for every ground canonical TRS $\mathcal{R}$

Proof

we prove $\#\,\mathsf{E}(\mathcal{R}) \leqslant \mathsf{bound}(\mathcal{R})$ by induction on $\#\mathcal{R}$

- $\mathcal{R} = \varnothing \quad \Longrightarrow \quad \#\,\mathsf{E}(\mathcal{R}) = \#\{\varnothing\} = 1 = \mathsf{bound}(\mathcal{R})$
- suppose $\mathcal{R} \neq \varnothing$ and let $t \in \mathsf{RHS}(\mathcal{R})$
- $\mathcal{R} = \mathcal{S} \uplus \mathcal{T}_{\to t}$ where $T = \{t\} \cup \{\ell \mid \ell \to t \in \mathcal{R}\}$
- $\#\,\mathsf{E}(\mathcal{R}) \leqslant \#\,T \times \#\,\mathsf{E}(\mathcal{S})$

$$\mathrm{bound}(\mathcal{R}) = \prod_{v \in \mathrm{RHS}(\mathcal{R})} \big(1 + \#\{\ell \to r \in \mathcal{R} \mid r = v\}\big)$$

Corollary

$\#\,\mathrm{CaP}(\mathcal{R}) \leqslant \mathrm{bound}(\mathcal{R})$ for every ground canonical TRS $\mathcal{R}$

Proof

we prove $\#\,\mathrm{E}(\mathcal{R}) \leqslant \mathrm{bound}(\mathcal{R})$ by induction on $\#\,\mathcal{R}$

- $\mathcal{R} = \varnothing \quad \Longrightarrow \quad \#\,\mathrm{E}(\mathcal{R}) = \#\{\varnothing\} = 1 = \mathrm{bound}(\mathcal{R})$
- suppose $\mathcal{R} \neq \varnothing$ and let $t \in \mathrm{RHS}(\mathcal{R})$
- $\mathcal{R} = \mathcal{S} \uplus \mathcal{T}_{\to t}$ where $T = \{t\} \cup \{\ell \mid \ell \to t \in \mathcal{R}\}$
- $\#\,\mathrm{E}(\mathcal{R}) \leqslant \#\,T \times \#\,\mathrm{E}(\mathcal{S}) \leqslant \#\,T \times \mathrm{bound}(\mathcal{S})$

$$\mathsf{bound}(\mathcal{R}) = \prod_{v \in \mathsf{RHS}(\mathcal{R})} \big(1 + \#\{\ell \to r \in \mathcal{R} \mid r = v\}\big)$$

Corollary

$\#\,\mathsf{CaP}(\mathcal{R}) \leqslant \mathsf{bound}(\mathcal{R})$ for every ground canonical TRS $\mathcal{R}$

Proof

we prove $\#\,\mathsf{E}(\mathcal{R}) \leqslant \mathsf{bound}(\mathcal{R})$ by induction on $\#\,\mathcal{R}$

- $\mathcal{R} = \varnothing \quad \Longrightarrow \quad \#\,\mathsf{E}(\mathcal{R}) = \#\{\varnothing\} = 1 = \mathsf{bound}(\mathcal{R})$
- suppose $\mathcal{R} \neq \varnothing$ and let $t \in \mathsf{RHS}(\mathcal{R})$
- $\mathcal{R} = \mathcal{S} \uplus \mathcal{T}_{\to t}$ where $T = \{t\} \cup \{\ell \mid \ell \to t \in \mathcal{R}\}$
- $\#\,\mathsf{E}(\mathcal{R}) \leqslant \#\,T \times \#\,\mathsf{E}(\mathcal{S}) \leqslant \#\,T \times \mathsf{bound}(\mathcal{S})$
- $\mathsf{bound}(\mathcal{R}) = (1 + \#\,\mathcal{T}_{\to t}) \times \mathsf{bound}(\mathcal{S})$

$$\mathsf{bound}(\mathcal{R}) = \prod_{v \in \mathsf{RHS}(\mathcal{R})} \big(1 + \#\{\ell \to r \in \mathcal{R} \mid r = v\}\big)$$

Corollary

$\#\,\mathsf{CaP}(\mathcal{R}) \leqslant \mathsf{bound}(\mathcal{R})$ for every ground canonical TRS $\mathcal{R}$

Proof

we prove $\#\,\mathsf{E}(\mathcal{R}) \leqslant \mathsf{bound}(\mathcal{R})$ by induction on $\#\,\mathcal{R}$

- $\mathcal{R} = \varnothing \quad \Longrightarrow \quad \#\,\mathsf{E}(\mathcal{R}) = \#\{\varnothing\} = 1 = \mathsf{bound}(\mathcal{R})$
- suppose $\mathcal{R} \neq \varnothing$ and let $t \in \mathsf{RHS}(\mathcal{R})$
- $\mathcal{R} = \mathcal{S} \uplus \mathcal{T}_{\to t}$ where $T = \{t\} \cup \{\ell \mid \ell \to t \in \mathcal{R}\}$
- $\#\,\mathsf{E}(\mathcal{R}) \leqslant \#\,T \times \#\,\mathsf{E}(\mathcal{S}) \leqslant \#\,T \times \mathsf{bound}(\mathcal{S})$
- $\mathsf{bound}(\mathcal{R}) = (1 + \#\,\mathcal{T}_{\to t}) \times \mathsf{bound}(\mathcal{S}) = \#\,T \times \mathsf{bound}(\mathcal{S})$

$$\mathrm{bound}(\mathcal{R}) = \prod_{v \in \mathrm{RHS}(\mathcal{R})} \big(1 + \#\{\ell \to r \in \mathcal{R} \mid r = v\}\big)$$

Corollary

$\#\,\mathrm{CaP}(\mathcal{R}) \leqslant \mathrm{bound}(\mathcal{R})$ for every ground canonical TRS $\mathcal{R}$

Proof

we prove $\#\,\mathrm{E}(\mathcal{R}) \leqslant \mathrm{bound}(\mathcal{R})$ by induction on $\#\,\mathcal{R}$

- $\mathcal{R} = \varnothing \quad \Longrightarrow \quad \#\,\mathrm{E}(\mathcal{R}) = \#\{\varnothing\} = 1 = \mathrm{bound}(\mathcal{R})$
- suppose $\mathcal{R} \neq \varnothing$ and let $t \in \mathrm{RHS}(\mathcal{R})$
- $\mathcal{R} = \mathcal{S} \uplus \mathcal{T}_{\to t}$ where $T = \{t\} \cup \{\ell \mid \ell \to t \in \mathcal{R}\}$
- $\#\,\mathrm{E}(\mathcal{R}) \leqslant \#\,T \times \#\,\mathrm{E}(\mathcal{S}) \leqslant \#\,T \times \mathrm{bound}(\mathcal{S})$
- $\mathrm{bound}(\mathcal{R}) = (1 + \#\,\mathcal{T}_{\to t}) \times \mathrm{bound}(\mathcal{S}) = \#\,T \times \mathrm{bound}(\mathcal{S})$
- $\#\,\mathrm{E}(\mathcal{R}) \leqslant \mathrm{bound}(\mathcal{R})$

Outline

Conclusion

- recursive function to enumerate equivalent canonical ground TRSs

Conclusion

- recursive function to enumerate equivalent canonical ground TRSs
- complicated correctness proof

Conclusion

- recursive function to enumerate equivalent canonical ground TRSs
- complicated correctness proof
- 2^k upper bound on number of different canonical presentations for ground ESs

Conclusion

- recursive function to enumerate equivalent canonical ground TRSs
- complicated correctness proof
- 2^k upper bound on number of different canonical presentations for ground ESs

Future Work

- formalisation

Conclusion

- recursive function to enumerate equivalent canonical ground TRSs
- complicated correctness proof
- 2^k upper bound on number of different canonical presentations for ground ESs

Future Work

- formalisation
- AC canonical presentations

Postdoc Position in Innsbruck (3 years)

- available from 1 October 2026 (automated reasoning, term rewriting, verification)
- 4 hours of teaching per week during semester
- application period: July 31 – August 20 MIP−15908

`https://lfuonline.uibk.ac.at/public/karriereportal.home`

Postdoc Position in Innsbruck (3 years)

- available from 1 October 2026 (automated reasoning, term rewriting, verification)
- 4 hours of teaching per week during semester
- application period: July 31 – August 20   MIP–15908

`https://lfuonline.uibk.ac.at/public/karriereportal.home`