Confluence of Orthogonal DPRSs

Johannes Niederhauser and Aart Middeldorp

University of Innsbruck

Outline

1. **Introduction**
2. **Deterministic Higher–Order Pattern Rewrite Systems**
3. **Critical Pairs**
4. **Orthogonal DPRSs**
5. **Conclusion**

Definition (Miller 1991)

pattern is term in β–normal form such that arguments of free variables are η–equivalent to distinct bound variables

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Example

$$x, y.F(y, x) \quad x.F(x, x) \quad x.F(\mathsf{c}, x) \quad x.F(\mathsf{f}(x), x) \quad x, y.F(\mathsf{f}(x), \mathsf{f}(y))$$

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term in β–normal form is deterministic higher-order pattern (DHP) if for all subterms $\overline{x}_n.Y(\overline{s}_m)$

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Example (deterministic higher–order patterns)

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unification problem for higher−order patterns is decidable

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Theorem (Nipkow 1993)

orthogonal PRSs are confluent

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Example (Yokoyama & Hu & Takeichi 2003)

- $\mathsf{f} : \mathsf{a} \to \mathsf{a} \quad M, N : (\mathsf{a}, \mathsf{a}) \to \mathsf{a}$
- terms $x, y.M(\mathsf{f}(x), \mathsf{f}(y))$ and $x, y.\mathsf{f}(N(y, x))$ admit three (incomparable) unifiers:

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- $\mathsf{f} : \mathsf{a} \to \mathsf{a} \quad M, N : (\mathsf{a}, \mathsf{a}) \to \mathsf{a} \quad Z : (a, a) \to a$
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Remarks

- we present sound and complete unification procedure for DHPs at IJCAR 2026

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- minimal complete sets of unifiers may be infinite

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Example

- $\mathsf{case} : (\mathsf{u}, \mathsf{a} \to \mathsf{i}, \mathsf{b} \to \mathsf{i}) \to \mathsf{i} \qquad \mathsf{inl} : \mathsf{a} \to \mathsf{u} \qquad \mathsf{inr} : \mathsf{b} \to \mathsf{u}$
- DPRS $\mathcal{R}$

$$\begin{aligned}
\mathsf{case}(\mathsf{inl}(X), w.F(w), w.G(w)) &\to F(X) \\
\mathsf{case}(\mathsf{inr}(X), w.F(w), w.G(w)) &\to G(X) \\
\mathsf{case}(Z, w.H(\mathsf{inl}(w)), w.H(\mathsf{inr}(w))) &\to H(Z)
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- COPS #769
- 17 out of 124 HRSs in COPS are DPRSs

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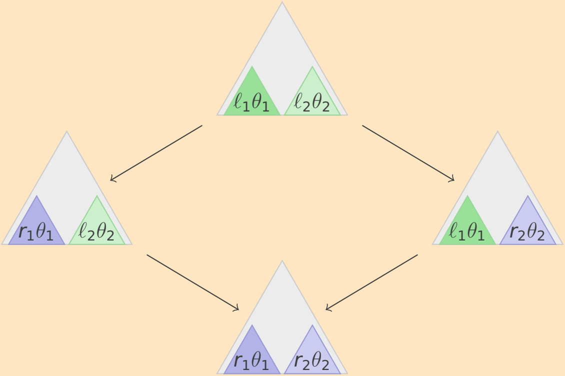
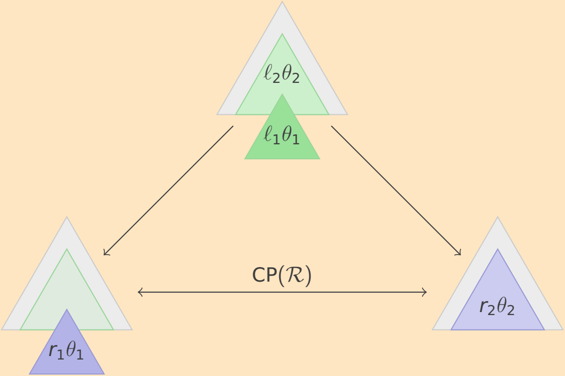
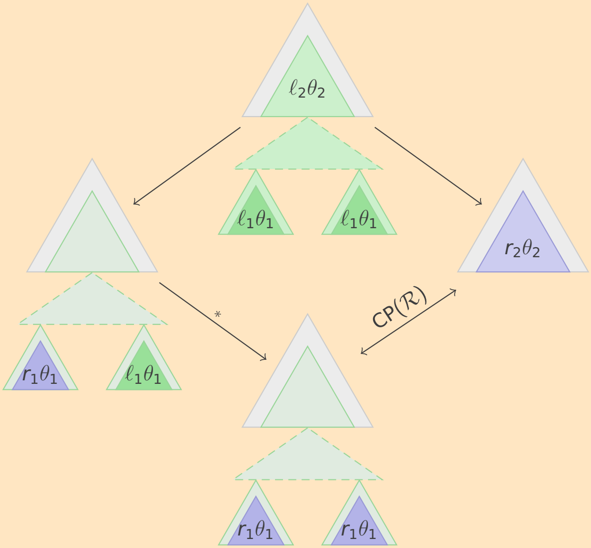
Parallel redexes



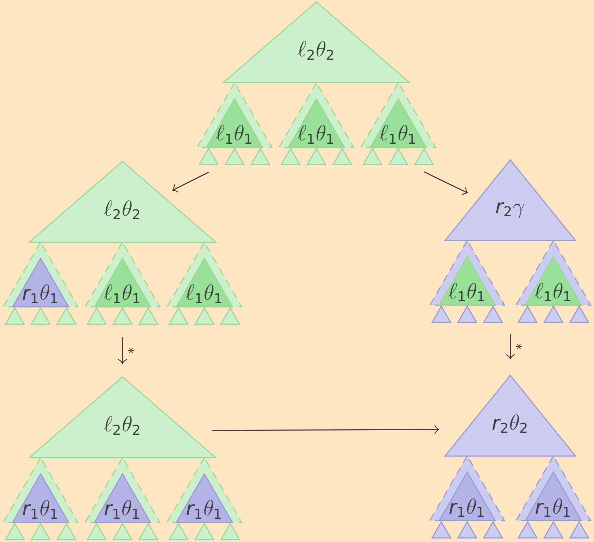
Standard Critical Overlap



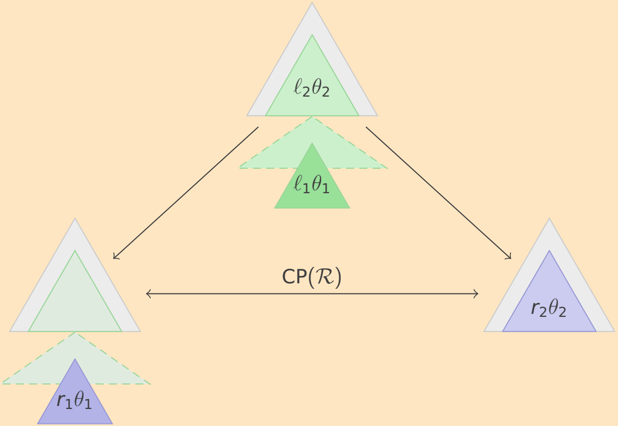
Variable Argument Overlap



Trivial Variable Overlap



Critical Variable Overlap



Theorem

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Remarks

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- more pressing issue: critical pairs often do not produce DHPs
- implementation available in prototype hrstk

Outline

1. Introduction
2. Deterministic Higher–Order Pattern Rewrite Systems
3. Critical Pairs
4. **Orthogonal DPRSs**
5. Conclusion

Definition

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Theorem

orthogonal DPRSs are confluent

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